Upper bounds for the formula size of the majority function *

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Abstract

It is shown that the counting function of n Boolean variables can be implemented with the formulae of size $O(n^{3.06})$ over the basis of all 2-input Boolean functions and of size $O(n^{4.54})$ over the standard basis. The same bounds follow for the complexity of any threshold symmetric function of n variables and particularly for the majority function. Any bit of the product of binary numbers of length n can be computed by formulae of size $O(n^{4.06})$ or $O(n^{5.54})$ depending on basis. Incidentally the bounds $O(n^{3.23})$ and $O(n^{4.82})$ on the formula size of any symmetric function of n variables with respect to the basis are obtained.

1 Introduction

We consider the complexity of implementation of symmetric Boolean functions with formulae over the basis B_2 of all binary Boolean functions and over the standard basis $B_0 = \{\wedge, \vee, \bar{}\}$. All necessary notions of formulae and complexity $L_B(f)$ of implementation of function f with formulae over basis B one can find in [1, 4].

The best known results on the complexity and the depth of implementation of symmetric functions with either circuits or formulae over complete bases depend on the efficient implementation of the counting function $C_n(x_1, \ldots, x_n)$ calculating the sum of Boolean variables $x_1, \ldots, x_n$. Reduction to the computation of C_n is a way of minimization of the depth and the complexity of formulae for multiplication of binary numbers.

*Research supported in part by RFBR, grants 11–01–00508, 11–01–00792, and OMN RAS "Algebraic and combinatorial methods of mathematical cybernetics and information systems of new generation" program (project "Problems of optimal synthesis of control systems").

In its turn, efficient circuits and formulae for the counting function can be built of CSA-units.[1] (k,l)-CSA of width 1 implements a Boolean function $(x_1,\ldots,x_k)\to(y_1,\ldots,y_l)$ according to condition $\sum a_i x_i = \sum b_j x_j$, where $k>l$ and constants a_i, b_j are the integer powers of two. (k,l)-CSA of arbitrary width can be composed of parallel copies of width 1 CSA's. It allows to reduce addition of k numbers to addition of l numbers.

With the use of appropriate CSA's and method [6] the bounds $L_{B_2}(C_n) = O(n^{3.13})$, $L_{B_0}(C_n) = O(n^{4.57})$ were obtained in [7] improving upon preceding known results $L_{B_2}(C_n) = O(n^{3.32})$ [8] and $L_{B_0}(C_n) = O(n^{4.62})$ [2] (some earlier results see in [6]).

In the recent paper [3] it was built a new CSA (called MDFA) which allows to implement the function C_n with a circuit over B_2 of the best known complexity $4.5n$ (improving the old previous bound $5n$). Efficient use of the CSA [3] implies encoding of some pairs of bits u, v in the form $(u\oplus v, v)$ (such encoding was introduced in [9]). A benefit in complexity is due to (a) one can compute a pair $(u\oplus v, v)$ not harder than (u,v), (b) following computations need $u\oplus v$ rather than u — that's why one Boolean addition can be saved.

Note that by the above reasons one can expect MDFA to be efficient for implementation with formulae. Indeed, MDFA allows shorter formulae for $u\oplus v$ than formulae for u and v together.

This observation was exploited implicitly in the construction of the formula efficient CSA in [7]. The CSA contains MDFA with its outputs connected to the standard $(3,2)$-CSA FA_3. What makes the CSA [7] more efficient than the standard $(3,2)$-CSA is exactly intermediate $(u\oplus v, v)$ encoding (the $(3,2)$-CSA alone leads to the bound $L_{B_2}(C_n) = O(n^{3.21})$, see [6]).

Moreover, as far as MDFA saves circuit complexity, CSA [7] also does it: its circuit complexity is 14.[2] Thus, the CSA allows to implement C_n with $(14/3)n$ circuit complexity. So, it was possible to overcome the $5n$ barrier in the 90-es.

It is natural to conclude that the CSA [7] does not use MDFA optimally to construct shorter formulae. We will show below that one can obtain the bound $L_{B_2}(C_n) = O(n^{3.06})$ via more independent way of exploiting MDFA. However, proposed method is also not optimal.

An analogous idea works in the case of basis B_0. In the case one can try monotone encoding $(uv,\ u\vee v)$ of a pair of bits u and v. Such encoding makes the most significant outputs of a CSA to be monotone functions of inputs (in fact, these functions are threshold if all inputs are of the same significance). It can be reasonable since the known CSA's over B_0 have non-monotone

[1] CSA is abbreviature for Carry Save Adder.

[2] Formula for the circuit implementation slightly differs from the shortest one.

outputs to be most difficult for implementation.

We will describe below $(5,3)$-CSA SFA_5, an analogue of MDFA for monotone encoding. With the use of it the bound $L_{B_0}(C_n) = O(n^{4.55})$ is rather simple to obtain. Slightly better bound $L_{B_0}(C_n) = O(n^{4.54})$ follows from the more complicated construction based on the $(7,3)$-CSA [2] and monotone encoding of triples of bits.

Let S_n denote the class of symmetric Boolean functions of n variables. New upper bounds for majority function does not provide automatic reduction of the size of formulae implementing functions from S_n due to the fact that the known methods [2, 5] limit the efficiency of CSA's with inputs and outputs of multiple types.

However, it is not hard to aggregate several CSA's with non-standard encoding of bits into single CSA with the standard encoding and apply the method [5] to obtain bounds $L_{B_2}(S_n) = O(n^{3.23})$, $L_{B_0}(S_n) = O(n^{4.82})$ improving earlier results $L_{B_2}(S_n) = O(n^{3.30})$ [5], $L_{B_2}(S_n) = O(n^{3.37})$ [8], $L_{B_0}(S_n) = O(n^{4.85})$ [5], $L_{B_0}(S_n) = O(n^{4.93})$ [2].

It is worth noting that basic CSA's in the present paper are structurally similar or in any case are not more complicated than the CSA's in preceding papers. So, the improvement in complexity bounds is entirely a result of exploiting the idea of alternative encoding of bits.

2 Formulae over B_2

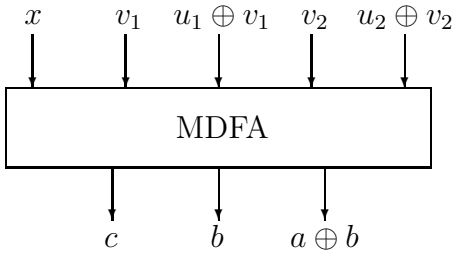

Figure 1: MDFA block-diagram

Fig. 1 shows a block-diagram of MDFA. Functional definition and formulae to compute outputs are given below.

$$2(a+b)+c = x+u_1+u_2+v_1+v_2.$$

$$\begin{array}{c} c = x \oplus (u_1 \oplus v_1) \oplus (u_2 \oplus v_2), \qquad b = (x \oplus v_1)(u_1 \oplus v_1) \oplus v_1, \\ (a \oplus b) = ((x \oplus v_1) \vee (u_1 \oplus v_1)) \oplus (x \oplus (u_1 \oplus v_1) \oplus v_2)\overline{(u_2 \oplus v_2)}. \end{array} \tag{1}$$

To obtain $O(n^{3.06})$ complexity bound one can use CSA shown in Fig. 2. It contains two isolated MDFA's, which are identical up to encoding of a pair of inputs.

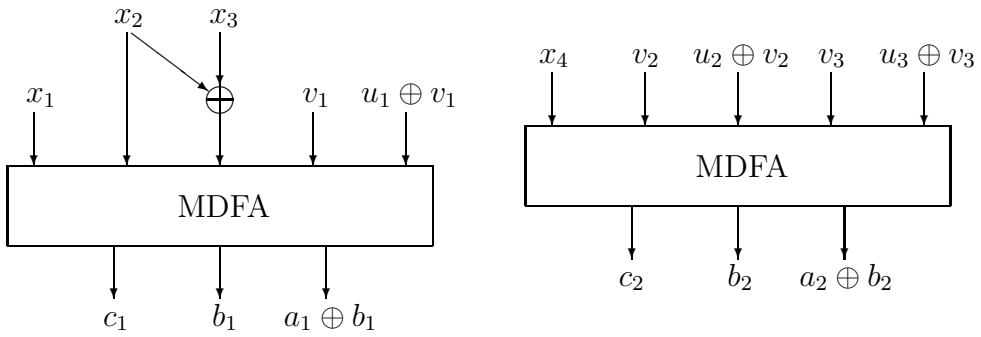

Figure 2: The main CSA

The CSA has inputs and outputs of two types: standard bits and pairs of bits encoded as $(u \oplus v, v)$.

Consider the size of formulae implementing inputs and outputs of the first type. Denote it by X_i for inputs x_i and by C_i for outputs c_i. To deal with inputs and outputs of the second type $(u \oplus v, v)$ introduce a quantity $\max\{V, U^+/\alpha\}$ where V, U^+ is the size (or an upper estimate of the size) of formulae implementing v and $u \oplus v$ respectively, α is a parameter to be chosen later. Denote this quantity by U_i for inputs $(u_i \oplus v_i, v_i)$ and by A_i for outputs $(a_i \oplus b_i, b_i)$.

According to (1), the following inequalities hold:

$$\begin{gathered} C_1 \le X_1 + X_2 + X_3 + \alpha U_1, \\ C_2 \le X_4 + \alpha U_2 + \alpha U_3, \\ A_1 \le \max\left\{X_1 + X_2 + 3X_3,\ \tfrac{2}{\alpha}X_1 + \tfrac{2}{\alpha}X_2 + \tfrac{3}{\alpha}X_3 + \tfrac{\alpha+1}{\alpha}U_1\right\}, \\ A_2 \le \max\left\{X_4 + (\alpha+2)U_2,\ \tfrac{2}{\alpha}X_4 + \tfrac{2\alpha+1}{\alpha}U_2 + \tfrac{\alpha+1}{\alpha}U_3\right\}. \end{gathered} \tag{2}$$

It follows from [6, 7] that if

$$\begin{gathered} X_1^p + X_2^p + X_3^p + X_4^p - C_1^p - C_2^p > 0, \\ U_1^p + U_2^p + U_3^p - A_1^p - A_2^p > 0, \end{gathered} \tag{3}$$

for some $p > 0$, some $X_i > 0$ and $U_i > 0$ (and also for some $\alpha > 0$) then $L_{B_2}(C_n) = O\left(n^{1/p+o(1)}\right)$, see also Appendix. Use upper bounds (2) instead of C_i and A_i to check that inequalities (3) hold when $\alpha = 2.906$, $p = 0.327781$, $X_1 = X_2 = 1$, $X_3 = 0.5149081$, $X_4 = 1.9198088$, $U_1 = 1.2176395$, $U_2 = 1.0031176$, $U_3 = 2.3573055$. Consequently, $L_{B_2}(C_n) = O(n^{3.0509})$.

To obtain tighter estimates of the efficiency of MDFA one can split formally the second type of encoding into several types with different values of α (say, uniformly distributed in some segment) and consider a set of MDFA's with inputs and outputs of all possible types. However, the implied calculation looks rather laborious if not to exploit some additional considerations. This observation is already involved partly in the construction of fig. 2: ratio of sizes of formulae for $x_2 \oplus x_3$ and x_2 differs from α.

To estimate the complexity of a symmetric function consider the following sequence of CSA's with standard encoding of bits. In the proposed sequence the m-th CSA contains m MDFA's connected in a chain and an outer CSA FA_3 (outputs of each MDFA are connected to the inputs v_2, $u_2 \oplus v_2$ of the next MDFA in notation of fig. 1). The first CSA in the sequence ($m = 1$) is the CSA from [7]. For $m = 4$ we get $(15, 6)$-CSA, which, if taken independently, allows to implement C_n with the formula of size $O(n^{3.089})$.[3]

With the use of this $(15, 6)$-CSA and method [5] one can implement a k-th significant bit of C_n with complexity $O(n^{2.2285} \cdot 2^k)$. Thus, the bound $L_{B_2}(S_n) = O(n^{3.2285})$ follows, see Appendix for proof.

3 Formulae over B_0

The present section includes two examples of CSA's which are efficient for constructing formulae for C_n over B_0. The first CSA is shown in fig. 3. Functional definition of the basic CSA SFA_5 (Sorting Full Adder) is similar to that of MDFA. Outputs are implemented by formulae:

$$\begin{gathered} c = (x_1 \oplus (u_1 \oplus v_1)) \oplus (u_2 \oplus v_2) = \psi\overline{\chi} \vee \overline{\psi}\chi, \\ \psi = x_1 \left((u_1 v_1) \vee \overline{(u_1 \vee v_1)} \right) \vee \overline{x_1}\,\overline{(u_1 v_1)}(u_1 \vee v_1), \quad \chi = \overline{(u_2 v_2)}(u_2 \vee v_2), \\ a_1 b_1 = T_5^4(x_1, u_1, v_1, u_2, v_2) = \bigvee_{i+j=4} T_3^i(x_1, u_1, v_1) T_2^j(u_2, v_2) = \\ = (x_1(u_1 \vee v_1) \vee (u_1 v_1))(u_2 v_2) \vee x_1(u_1 v_1)(u_2 \vee v_2), \end{gathered} \tag{4}$$

[3]Vector of sizes of outputs of the CSA can be produced from the vector of sizes of inputs via multiplication by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 & 1 & 2 & 3 & 3 & 6 \\ 0 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 6 & 1 & 2 & 3 & 3 & 6 \\ 1 & 2 & 2 & 3 & 3 & 3 & 6 & 3 & 3 & 6 & 1 & 2 & 3 & 3 & 6 \\ 1 & 4 & 4 & 9 & 3 & 3 & 6 & 3 & 3 & 6 & 1 & 2 & 3 & 3 & 6 \end{pmatrix}$$

where T_n^k is the threshold monotone function of n variables with the threshold k; output $a_1 \vee b_1 = T_5^2(x_1, u_1, v_1, u_2, v_2)$ is a function dual to $a_1 b_1$, so it can be implemented with the dual formula.

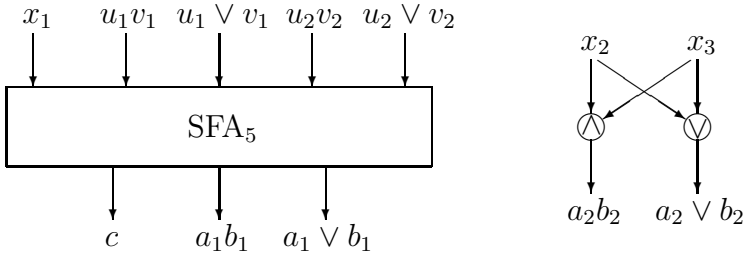

Figure 3: Block-diagram of the first CSA

By analogy with the previous section for inputs x_i, $(u_iv_i,\ u_i \vee v_i)$ and outputs c, $(a_ib_i,\ a_i \vee b_i)$ consider quantities X_i, U_i, C, A_i respectively, which correspond to the size of formulae implementing x_i, u_iv_i (or $u_i \vee v_i$ — here formulae for components in a pair have equal size), c, a_ib_i.

According to (4) the following inequalities hold:

$$C \le 4X_1 + 8U_1 + 4U_2, \quad A_1 \le 2X_1 + 3U_1 + 2U_2, \quad A_2 \le X_2 + X_3. \qquad (5)$$

One can easily check that the condition

$$X_1^p + X_2^p + X_3^p - C^p > 0 \qquad U_1^p + U_2^p - A_1^p - A_2^p > 0$$

is satisfied for $p = 0.219978$, $X_1 = 1$, $X_2 = X_3 = 0.031702$, $U_1 = 1.018913$, $U_2 = 2$. As a consequence, $L_{B_0}(C_n) = O(n^{4.546})$.

In the second example we encode triples of bits u, v, w as an ordered triple $\widetilde{s} = (s',\ s'',\ s''')$, supplemented with a sum $s^{\oplus} = u \oplus v \oplus w$. Let us give formulae to compute the code components:

$$s' = \min\{u,\ v,\ w\} = T_3^1(u, v, w) = u \vee v \vee w,$$

$$s'' = T_3^2(u, v, w) = (u \vee v)w \vee uv,$$

$$s''' = \max\{u,\ v,\ w\} = T_3^3(u, v, w) = uvw.$$

Note that s'' and $s^{\oplus}$ are exact bits representing the sum $u + v + w$.

CSA of the second example (see fig. 4) contains two $(7, 4)$-CSA's SFA_7 and SFA'_7, differing in encoding of a triple of inputs.

SFA_7 and SFA'_7 are functionally defined by equalities:

$$c_1 + 2(q'_1 + q''_1 + q'''_1) = x_1 + s'_1 + s''_1 + s'''_1 + s'_2 + s''_2 + s'''_2,$$

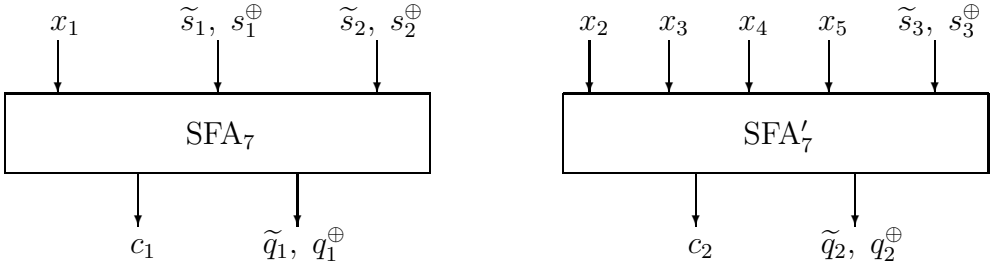

Figure 4: Block-diagram of the second CSA

$$c_2 + 2(q'_2 + q''_2 + q'''_2) = x_2 + x_3 + x_4 + x_5 + s'_3 + s''_3 + s'''_3.$$

Outputs are implemented with formulae, structurally similar to those for the $(7,3)$-CSA [2]:

$$\begin{gathered}
c_1 = (x_1 \oplus s_1^{\oplus}) \oplus s_2^{\oplus} = s_2^{\oplus}\overline{\left(x_1\overline{s_1^{\oplus}} \vee \overline{x}_1 s_1^{\oplus}\right)} \vee \overline{s_2^{\oplus}}\left(x_1\overline{s_1^{\oplus}} \vee \overline{x}_1 s_1^{\oplus}\right), \\
c_2 = (x_2 \oplus x_3 \oplus x_4 \oplus x_5) \oplus s_3^{\oplus} = \psi\overline{s_3^{\oplus}} \vee \overline{\psi} s_3^{\oplus}, \\
\psi = (x_2\overline{x}_3 \vee \overline{x}_2 x_3)(x_4 x_5 \vee \overline{x}_4\overline{x}_5) \vee (x_2 x_3 \vee \overline{x}_2\overline{x}_3)(x_4\overline{x}_5 \vee \overline{x}_4 x_5), \\
q'_1 = T_7^2(x_1, \widetilde{s}_1, \widetilde{s}_2) = T_4^1(x_1, \widetilde{s}_1)s'_2 \vee T_4^2(x_1, \widetilde{s}_1) \vee s''_2, \\
q''_1 = T_7^4(x_1, \widetilde{s}_1, \widetilde{s}_2) = \bigvee_{i+j=4} T_4^i(x_1, \widetilde{s}_1)T_3^j(\widetilde{s}_2), \\
q_1^{\oplus} = \overline{s'_2}\,T_4^2(x_1, \widetilde{s}_1)\overline{T_4^4(x_1, \widetilde{s}_1)} \vee s'_2\overline{s''_2}\,T_4^1(x_1, \widetilde{s}_1)\overline{T_4^3(x_1, \widetilde{s}_1)} \vee \\
\vee\ s''_2\overline{s'''_2}\left(T_4^4(x_1, \widetilde{s}_1) \vee \overline{T_4^2(x_1, \widetilde{s}_1)}\right) \vee s'''_2\left(T_4^3(x_1, \widetilde{s}_1) \vee \overline{T_4^1(x_1, \widetilde{s}_1)}\right).
\end{gathered} \tag{6}$$

Formula for $q'''_1 = T_7^6(x_1, \widetilde{s}_1, \widetilde{s}_2)$ is dual to that for q'_1 up to substitution s'_i by s'''_i. Formulae for $\widetilde{q}_2$ and $q_2^{\oplus}$ coincide to those for $\widetilde{q}_1$ and $q_1^{\oplus}$ up to implementation of the threshold funcion $T_4(x_2, x_3, x_4, x_5)$.

Threshold functions $T_4^i(y_1, y_2, y_3, y_4) = T_4^i(y_1, \widetilde{s})$ with three of variables allowing multiple encoding can be implemented with formulae:

$$\begin{gathered}
T_4^1 = y_1 \vee y_2 \vee y_3 \vee y_4 = y_1 \vee s', \\
T_4^2 = (y_1 \vee y_2)(y_3 \vee y_4) \vee y_1 y_2 \vee y_3 y_4 = y_1 s' \vee s''.
\end{gathered} \tag{7}$$

Formulae for T_4^3 and T_4^4 are obtained in dual way with substitution s' by s'''.

According to the construction, formulae for q'_1 and q'''_1 (q'_2 and q'''_2) have the equal size if the same holds for inputs s'_i, s'''_i. Furthermore, the size of formula for $q_i^{\oplus}$ is twice as much as that for q''_i.

As above, denote the complexity of formulae implementing inputs x_i and outputs c_i by X_i and C_i respectively. For inputs $\widetilde{s}_i$ consider the quantity $S_i = \max\{S'_i,\ S''_i/\alpha\}$, where S'_i and S''_i characterize the size (to be more exact, an upper bound for the size) of formulae implementing s'_i and s''_i. Define quantities Q_i for outputs $\widetilde{q}_i$ analogously.

From (6) and (7) the inequalities follow:

$$\begin{aligned} C_1 &\le 4X_1 + 8\alpha S_1 + 4\alpha S_2, \\ C_2 &\le 8(X_2 + X_3 + X_4 + X_5) + 4\alpha S_3, \\ Q_1 &\le \max\left\{2X_1 + (\alpha+2)S_1 + (\alpha+1)S_2,\ \tfrac{4}{\alpha}X_1 + \tfrac{2\alpha+4}{\alpha}S_1 + \tfrac{\alpha+2}{\alpha}S_2\right\}, \\ Q_2 &\le \max\left\{3(X_2 + X_3 + X_4 + X_5) + (\alpha+1)S_3,\right. \\ &\qquad \left.\tfrac{6}{\alpha}(X_2 + X_3 + X_4 + X_5) + \tfrac{\alpha+2}{\alpha}S_3\right\}. \end{aligned} \tag{8}$$

With the use of (8) it can be checked that the condition

$$X_1^p + X_2^p + X_3^p + X_4^p + X_5^p - C_1^p - C_2^p > 0 \qquad S_1^p + S_2^p + S_3^p - Q_1^p - Q_2^p > 0$$

holds for $\alpha = 1.6782$, $p = 0.2204718$, $X_1 = 1$, $X_2 = X_3 = X_4 = X_5 = 0.3569540333$, $S_1 = 1.1282983248$, $S_2 = 2.424317629$, $S_3 = 1.6884745179$. Therefore, $L_{B_0}(C_n) = O(n^{4.5358})$.

The last bound probably can be improved even without constructing CSA's more complicated than SFA_5 and SFA_7, if one uses all three ways of encoding in unique CSA.

To estimate the complexity of an arbitrary symmetric function consider a $(17, 6)$-CSA with the standard encoding of inputs and outputs, containing SFA_7 and pair of SFA_5's in the bottom and $(7, 3)$-CSA similar to the CSA [2] at the top. Non-standard outputs of SFA's are connected to the inputs of $(7, 3)$-CSA.[4] One can verify that this CSA allows to implement C_n with complexity $O(n^{4.558})$ and a k-th significant bit of C_n with complexity $O(n^{3.8183} \cdot 2^k)$. So, the bound $L_{B_0}(S_n) = O(n^{4.8183})$ follows.

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$$\begin{pmatrix} 4 & 8 & 8 & 8 & 8 & 8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 8 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 8 & 8 \\ 12 & 16 & 16 & 24 & 24 & 24 & 24 & 16 & 16 & 16 & 24 & 24 & 16 & 16 & 16 & 24 & 24 \\ 14 & 20 & 20 & 24 & 24 & 24 & 24 & 24 & 24 & 24 & 36 & 36 & 24 & 24 & 24 & 36 & 36 \\ 7 & 10 & 10 & 12 & 12 & 12 & 12 & 12 & 12 & 12 & 18 & 18 & 12 & 12 & 12 & 18 & 18 \end{pmatrix}$$

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Appendix

To make the presentation complete we provide here a method of constructing formulae, which can be also found in [5, 6, 7].

1. Implementation of C_n.

Consider a CSA with inputs and outputs of t types of encoding. Let $x_{i,j}$ and $y_{i,j}$ denote inputs and outputs of j-th type. Let the size $Y_{k,l}$ of the formula implementing an output $y_{k,l}$ is a continuous, piecewise-linear and nondecreasing (with respect to each argument) function of sizes $X_{i,j}$ of the formulae implementing inputs $x_{i,j}$, where $X_{i,j}$, $Y_{k,l}$ take on arbitrary real non-negative values. Assume that if $Y_{k,l} < X_{i,j}$ then $y_{k,l}$ does not depend on $x_{i,j}$. Let the inequalities

$$\sum_i X_{i,j}^p - \sum_i Y_{i,j}^p > 0 \tag{9}$$

hold for some $p > 0$, some $X_{i,j} > 0$ and all $j = 1, \ldots, t$.

We are to show how one can built a formula of size $O\left(n^{1/p+o(1)}\right)$ to implement C_n.

Without loss of generality assume that $\min\{X_{i,j}\} = 1 < \min\{Y_{i,j}\}$. As Y's depend on X's continuously there exists such $\delta > 0$ that for any j inequality (9) remains true after the substitution $X_{i,j}$ and $Y_{i,j}$ by parameters $X'_{i,j} \in [X_{i,j} - \delta,\, X_{i,j}]$ and $Y'_{i,j} \in [Y_{i,j},\, Y_{i,j} + \delta]$. Then there exist (small enough) $\lambda > 1$ and $d^X_{i,j}, d^Y_{i,j} \in \mathbb{Z}$ such that$\lambda^{d^X_{i,j}/p} \in [X_{i,j} - \delta,\, X_{i,j}]$ and $\lambda^{d^Y_{i,j}/p} \in [Y_{i,j},\, Y_{i,j} + \delta]$ for all $i,\, j$. Consequently for any j the following inequality holds:

$$\sum_i \lambda^{d^X_{i,j}} - \sum_i \lambda^{d^Y_{i,j}} > 0.$$

Note that $\lambda^{d^X_{i,j}/p}$ is a lower bound for $X_{i,j}$ and $\lambda^{d^Y_{i,j}/p}$ is an upper bound for $Y_{i,j}$. Let us name a number $d^X_{i,j}$ (respectively $d^Y_{i,j}$) level of the input $x_{i,j}$ (output $y_{i,j}$). We can assume $\min\{d^X_{i,j}\} = 0$. Let $d = \max\{d^Y_{i,j}\}$.

Formula representing a bit of the function C_n can be constructed after the following pattern. The formula contains CSA's on different levels. Each CSA can receive either inputs of the formula, or outputs of other CSA's, or zero formulae as inputs. CSA on a level k receives inputs of j-th type on the levels $d^X_{i,j} + k$ and produces outputs of the same type on the levels $d^Y_{i,j} + k$. The formula receives its nonzero inputs (i.e. symbols of variables) on the level d and higher.

The formula is determined by the number $\left\lceil cn\lambda^{-k} \right\rceil$ of CSA's on each level k, $0 \le k \le \log_\lambda n$, where c is a constant to be defined later.

Let us estimate the number of inputs, including zeros, and the number of outputs of a type j in the formula. We will omit indices j in the argument below as it does not depend on j.

According to the construction, all outputs of the formula on the levels d and lower are zero. A total number of inputs (all zero) on the same levels is $O(n)$. Difference between the number of inputs and the number of outputs on a level k, $d \le k \le \log_\lambda n$, is

$$\sum_i \left\lceil cn\lambda^{d^X_i - k} \right\rceil - \sum_i \left\lceil cn\lambda^{d^Y_i - k} \right\rceil =$$

$$= cn\lambda^{-k} \left(\sum_i \lambda^{d^X_i} - \sum_i \lambda^{d^Y_i} \right) \pm O(1) = \Theta\left(n\lambda^{-k}\right) \pm O(1).$$

On the levels higher than $\log_\lambda n$ the formula receives and produces $O(1)$ inputs and outputs in total.

Hence, the formulae receives $\Theta(n)$ nonzero inputs and produces $O(\log n)$ nonzero outputs (of j-th type). One can choose c large enough to provide not less than n inputs for any j.

Consider the size of outputs. It follows from the definition of λ that inputs and outputs on level k are bounded above by $\lambda^{k/p}$. Thus the size of outputs is $\lambda^{(\log_\lambda n+O(1))/p} = O(n^{1/p})$.

To implement the C_n function one has to take $\lfloor \log_2 n \rfloor + 1$ parallel copies of the described pattern, zero some inputs and re-commutate appropriately inputs and outputs on each level. Final addition of $O(\log n)$ numbers can be implemented with an arbitrary polynomial-size formula. So, the overall size of the formulae for C_n is $O\left(n^{1/p}\log^{O(1)} n\right)$.

2. Formulae for symmetric functions.

Consider a CSA with standard encoding of inputs and outputs. Let $x_{s,i}$ and $y_{s,i}$ stand for inputs and outputs of s-th significant bit, $s \geq 0$. Let $X_{s,i}$ and $Y_{s,i}$ stand for the size of corresponding formulae. For any s define

$$a_s = \sum_i X_{s,i}^p - \sum_i Y_{s,i}^p,$$

where we suppose sums over empty set of indices to be zero. Let the inequalities

$$a_0 > 0, \qquad \sum_s a_s \nu^{-s} > 0. \tag{10}$$

hold for some p, $X_{s,i}$ and $\nu \geq 1$.

We will show that l-th significant bit of C_n can be implemented with a formula of size $O((\nu^l n)^{1/p+o(1)})$.

As above, choose an appropriate $\lambda > 1$ and approximate $X_{s,i}$ and $Y_{s,i}$ by integer powers of λ preserving (10) (denote the exponents by $d^X_{s,i}$, $d^Y_{s,i}$). Without loss of generality assume $\min\{d^X_{s,i}\} = 0$. Define $d = \max\{d^Y_{s,i}\}$.

Let CSA on the level k and of significance l receive inputs of significance $s + l$ on levels $d^X_{s,i} + k$ and produce outputs of significance $s + l$ on levels $d^Y_{s,i} + k$. Consider a formula containing $\lceil c\nu^l n\lambda^{-k} \rceil$ CSA's of significance l, $0 \leq l \leq \log_2 n + 1$, on a level k, $0 \leq k \leq \log_\lambda(\nu^l n)$. Nonzero inputs of the formula are received on the levels d and higher, all of significance 0.

We are to estimate the number of inputs and outputs of significance l on level k. If $d \leq k \leq \log_\lambda(\nu^l n)$, then difference between the number of inputs and the number of outputs is

$$\sum_{s,i} \left\lceil c\nu^{l-s} n\lambda^{d^X_{s,i}-k} \right\rceil - \sum_{s,i} \left\lceil c\nu^{l-s} n\lambda^{d^Y_{s,i}-k} \right\rceil =$$

$$= c\nu^l n\lambda^{-k} \sum_s a_s \nu^{-s} \pm O(1) = \Theta(\nu^l n\lambda^{-k}) \pm O(1).$$

On the levels higher than $\log_\lambda(\nu^l n)$ the formula receives and produces $O(1)$ inputs and outputs in total.

Therefore, the formula produces $O(\log n)$ outputs of any significance. A choice of large enough constant c provides at least n inputs of significance 0. Each output of significance l is implemented with a formula of size at most $\lambda^{(\log_\lambda(\nu^l n)+O(1))/p} = O((\nu^l n)^{1/p})$. Hence, l-th significant bit of C_n can be implemented with a formula of size $O((\nu^l n)^{1/p} \log^{O(1)} n)$.

Assuming $\nu \leq 2^p$ we obtain an upper bound $O(n^{1+1/p+o(1)})$ on the formula size complexity of the class S_n. The implied formulae are constructed simply via representation of a symmetric function as a function of the weight of its set of arguments and decomposition along (new) variables.