An explicit finite B_k -sequence

Igor S. Sergeev^{*}

Abstract

For any n and k, we provide an explicit (that is, computable in polynomial time) example of integer \mathcal{B}_k -sequence of size n consisting of elements bounded by $n^{k+o(k)}$.

dedicated to the memory of Vladimir Evgen'evich Alekseev (1943–2020)

Introduction. Recall that a set B in some commutative group is a \mathcal{B}_k -sequence if all k-element sums in B are different, that is, the equality

$$a_1 + \ldots + a_k = b_1 + \ldots + b_k, \qquad a_i, b_i \in B,$$

holds iff the multisets of summands coincide: $\{a_1, \ldots, a_k\} = \{b_1, \ldots, b_k\}.$

 \mathcal{B}_2 -sequences are also known as *Sidon sequences*. Very often, the notion of Sidon sequence stands as a synonym for \mathcal{B}_k -sequence in general.

Easy to check, if \mathbb{Z}_N contains a size- $n \mathcal{B}_k$ -sequence, then $N \ge \binom{n+k-1}{k}$. We want to consider only satisfactorily dense size- $n \mathcal{B}_k$ -sequences, say, for $N = (n+k)^{O(k)}$, avoiding trivial examples like $\{k, k^2, \ldots, k^n\}$ with exponentially large elements. Also, we interest in *explicit* constructions, that is, those that can be computed in polynomial time with respect to the binary size¹.

History. The most famous explicit examples of the optimal density integer Sidon sequences are: a size-(q+1) set in \mathbb{Z}_{q^2+q+1} due to J. Singer [9], a size-q set in \mathbb{Z}_{q^2-1} due to R. C. Bose [2], and a size-(p-1) set in \mathbb{Z}_{p^2-p} due to V. E. Alekseev [1]. Here p and q stay for any prime number and prime power, respectively. The latter set is attributed to I. Ruzsa [7] almost everywhere.

The classical example of a nearly dense-optimal \mathcal{B}_k -sequence was proposed by Bose and S. Chowla in [3]. Let us recall this construction that generalizes [2]. Let $GF(q) = \{\alpha_1, \ldots, \alpha_q\}$, and x be a primitive element in $GF(q^k)$. It can be easily verified that the set

$$D[q,k] = \{d_i \mid x^{d_i} = x + \alpha_i, \ 1 \le d_i < q^k\}$$

^{*}e-mail: isserg@gmail.com

¹That is, the length of the binary code representing the elements of the set.

is a size- $q \mathcal{B}_k$ -sequence in \mathbb{Z}_{q^k-1} .

There are known also a number of similar constructions including another \mathcal{B}_k -sequence from [3] generalizing [9]. H. Derksen [4] proposed even more general constructions considering quotient polynomial rings GF(q)[x]/(P(x)) instead of pure fields in the examples from [3]. C. A. Gómez Ruiz and C. A. Trujillo Solarte [5] extended an example [1] to \mathcal{B}_k -sequences in \mathbb{Z}_{p^k-p} .

Discussion. All these examples of \mathcal{B}_k -sequences may be considered explicit only for constant or extremely slowly growing k's with respect to n, since they imply computation of discrete logarithms in groups of generally non-smooth order. Indeed, probabilistic or greedy constructions that we haven't mentioned are even less explicit. It looks like we lack easily computable and dense enough examples of \mathcal{B}_k -sequences that could be useful in some specific situations, e.g. for proving explicit lower bounds in computational complexity [8]. Thus, we intend to close this gap.

We follow the general idea of previous constructions: computing an additive numeric \mathcal{B}_k -sequence as an image of some simple multiplicative \mathcal{B}_k sequence from an appropriate group. All we need to make computations easy is to choose a basic multiplicative group of smooth order. Note that in doing this, we will partially sacrifice the density.

Construction. Further, p_1, p_2, \ldots denote odd prime numbers written in growing order. Let $r = 1 + \lceil k \log p_n \rceil$. The set of odd numbers-residues from 1 to $2^r - 1$ constitutes the multiplicative group $\mathbb{Z}_{2^r}^*$ of the ring \mathbb{Z}_{2^r} . For $r \ge 3$, this group is a direct product of cyclic groups of orders 2 and 2^{r-2} , namely, $\mathbb{Z}_{2^r}^* \cong \langle -1 \rangle_2 \langle 5 \rangle_{2^{r-2}}$ with -1 and 5 being generating elements. Therefore, any odd number x has a unique representation $x \equiv (-1)^j \cdot 5^h \pmod{2^r}$, where $0 \le j \le 1$ and $0 \le h < 2^{r-2}$. For details, see e.g. [10].

Consider the number set

$$H[n,k] = \{h_i \mid p_i \equiv \pm 5^{h_i} \pmod{2^r}, \ 0 \le h_i < 2^{r-2}, \ i = 1, \dots, n\}.$$

Let us check that the given set is a \mathcal{B}_k -sequence in $\mathbb{Z}_{2^{r-2}}$. By the choice of r, for different tuples of indices $1 \leq i_1 \leq \ldots \leq i_k \leq n$, all numbers $\pm p_{i_1} \cdots p_{i_k}$ are different and do not exceed $2^{r-1}-1$ by absolute value. Hence, all residues $5^{h_{i_1}+\ldots+h_{i_k}} \pmod{2^r}$ are different, and all sums $h_{i_1}+\ldots+h_{i_k} \pmod{2^{r-2}}$ are different as well.

The set H[n, k] is not as dense as D[q, k] or similar constructions. Still, its density is satisfactorily in asymptotic sense: $2^{r-2} < p_n^k < (2n \log(n+2))^k$ due to the well-known facts about distribution of prime numbers, see e.g. [6].

We are left to confirm explicitness: that the set H[n,k] requires $(n+k)^{O(1)}$ time to be constructed. First, we need to obtain the list of prime

numbers. Second, we have to compute discrete logarithms² $\log_5(\pm p_i)$ in \mathbb{Z}_{2^r} . For the first part, we may use Eratosthenes sieve or any other known algorithm running in time $n^{O(1)}$. Discrete logarithm in the cyclic group of order 2^{r-2} may be computed trivially by $O(r^2)$ elementary arithmetic operations mostly consisting of squarings. Indeed, we may determine binary digits of the number $a = [a_{r-3}, \ldots, a_0]_2 = \log_5 x \pmod{2^r}$ sequentially as

$$a_0 = \log_{5^{2^{r-3}}} x^{2^{r-3}}, \quad a_1 = \log_{5^{2^{r-3}}} (5^{-a_0} x)^{2^{r-4}}, \quad \dots,$$

 $a_{r-3} = \log_{5^{2^{r-3}}} (5^{-2^{r-4} a_{r-4} - \dots - 2a_1 - a_0} x).$

Inner logarithms are performed in an order-2 subgroup with generating element $5^{2^{r-3}} \equiv 2^{r-1} + 1 \pmod{2^r}$ simply by comparing with 1 and $2^{r-1} + 1$. If both comparisons fail, then $x \notin \langle 5 \rangle_{2^{r-2}}$.

Notes. In the above example, we intentionally used as smooth order for the basic multiplicative group as possible. Instead, we can work in any ring \mathbb{Z}_{p^r} with an odd prime p. The multiplicative group $\mathbb{Z}_{p^r}^*$ has order $(p-1)p^{r-1}$ and it is cyclic. The case p = 3 is especially attractive, since there we have 2 as a generating element for the multiplicative group. With more care, we can consider residue rings of some other smooth orders.

The choice of prime numbers for a "factor base" is also changeable. Say, we can relax the condition of being prime to the condition of being pairwise prime. Though, this relaxation alone doesn't allow to substantially increase the density of the set.

Essentially, the present text in an excerpt from [8].

References

- Alekseev V. E. Two constructions of difference sets. Problems of cybernetics. Vol. 38. Moscow: Nauka, 1981, 259–262 (in Russian).
- [2] Bose R. C. An affine analogue of Singer's theorem. J. Indian Math. Soc. (N.S.) 1942. 6, 1–15.
- [3] Bose R. C., Chowla S. *Theorems in the additive theory of numbers*. Commentarii Mathematici Helvetici. 1962. **37**, 141—147.
- [4] Derksen H. Error-correcting codes and B_h -sequences. IEEE Trans. Inform. Theory. 2004. **50**(3), 476–485.
- [5] Gómez Ruiz C. A., Trujillo Solarte C. A. Una nueva construcción de conjuntos B_h modulares. Matemáticas: Enseñanza Universitaria. 2011. XIX(1), 53–62.

²Here, we don't resort to the commonly used notation $\operatorname{ind}_q x$.

- [6] Rosser J. B., Schoenfeld L. Approximate formulas for some functions of prime numbers. Illinois J. Math. 1962. 6, 64–94.
- [7] Ruzsa I. Z. Solving a linear equation in a set of integers. I. Acta Arithm. 1993. 65, 259–282.
- [8] Sergeev I. S. On the additive complexity of some number sequences. Submitted to J. Applied and Industrial Math.
- [9] Singer J. A theorem in finite projective geometry and some applications to number theory. Trans. Amer. Math. Soc. 1938. 43, 377–385.
- [10] Vinogradov I. M. Elements of number theory. NY: Dover, 1954.