Upper bounds for the monotone rank of the unique disjointness matrix

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Abstract

It is shown that the OR-rank (covering rank) of the $2^n \times 2^n$ unique disjointness matrix is $n^{O(1)}(3/2)^n$, hence the known lower bound 1.5^n turns out to be essentially tight. By the way, an upper bound 1.89^n is obtained for the SUM-rank (partition rank) of this matrix.

1 Introduction

The *unique disjointness matrix* U_n (often denoted as $UDISJ_n$) is a partially defined $2^n \times 2^n$ boolean matrix. Its rows and columns are numbered by subsets of $[n]$, and the entries are defined as

$$U_n[a,b] = \begin{cases} 1, & a \cap b = \varnothing \\ 0, & |a \cap b| = 1 \\ *, & |a \cap b| > 1 \end{cases}.$$

Further, when numbering the rows and columns of the matrix, in addition to subsets $a \subset [n]$, we will also use their characteristic boolean vectors of length n.

Although the unique disjointness matrix arises indirectly in Razborov's work [7] on a problem in the field of communication complexity, today results on the monotone rank of this matrix find applications in the theory of optimization of linear programs; see, e.g., [2, 6]. However, many problems in these two areas (communication complexity and linear program optimization) are closely related, are reduced to each other, or are studied using similar approaches; see, e.g., [4].

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The *rank* of a matrix A over a semiring $(S, +, \cdot)$ is defined as the minimum number t of terms in a decomposition

$$A = R_1 + \ldots + R_t, \qquad R_i = \begin{pmatrix} a_{i,1} \\ \vdots \\ a_{i,m} \end{pmatrix} \cdot \begin{pmatrix} b_{i,1} & \ldots & b_{i,n} \end{pmatrix}, \quad a_{i,j}, b_{i,k} \in S.$$

In the boolean case, R_i are all-ones submatrices of A; they are usually called (combinatorial) *rectangles*. For partially defined boolean matrices, a rectangle is conveniently defined as a submatrix without zeros.

Let $\mathbb{B} = \{0,\, 1\}$. The main types of monotone rank of a (boolean partially defined) matrix A:

— the rank $\mathrm{rk}_\vee A$ over the boolean semiring $(\mathbb{B}, \vee, \wedge)$, also denoted as OR-rank, the covering rank;

— the rank $\mathrm{rk}_{\mathbb{R}+} A$ over the real semiring $(\mathbb{R}_+, +, *)$, also known as the nonnegative rank;

— the rank $\mathrm{rk}_+ A$ over the integer semiring $(\mathbb{N}_{\geq 0}, +, *)$, also denoted as the SUM-rank, the partition rank.

These quantities obey the (obvious) inequalities

$$\mathrm{rk}_\vee A \leq \mathrm{rk}_{\mathbb{R}+} A \leq \mathrm{rk}_+ A.$$

Covering and partition ranks characterize the communication complexity of a matrix: the nondeterministic communication complexity of a matrix A is $\lceil \log_2 \mathrm{rk}_\vee A \rceil$, and the unambiguous nondeterministic communication complexity is $\lceil \log_2 \mathrm{rk}_+ A \rceil$. In optimization problems with linear constraints, the nonnegative real rank is studied. Due to the fundamental nature of the matrix, the question of its rank can be considered as being of independent interest.

Among the well-known completions of the matrix U_n are the Sierpiński matrix[1] S_n (the disjointness matrix) and the complement of the boolean version of the Sylvester matrix[2] H_n. These matrices have full OR-rank, $\mathrm{rk}_\vee S_n = \mathrm{rk}_\vee H_n = 2^n$.

As usual, the *weight* of a (partially defined) boolean matrix is defined as the number of ones in it. An elegant argument from [6] proves that any rectangle in U_n has weight $\leq 2^n$, so $\mathrm{rk}_\vee U_n \geq 1.5^n$, since the weight of the matrix is 3^n. A simple construction from [12] shows that $\mathrm{rk}_\vee U_n = O(\sqrt{3}^{\,n})$.

In [1], it was established that the method [12] yields an exact value for the OR-rank of U_n for $n \leq 6$. However, below we will show that the lower

[1] With entries $S_n[a, b] = (a \cap b = \varnothing)$.

[2] With entries $H_n[a, b] = |a \cap b| + 1 \bmod 2$.

bound [6] is in fact tight: $\mathrm{rk}_\vee U_n = n^{O(1)}(3/2)^n$. Note that this result echoes a similar estimate for the extension complexity of the matching polytope in [11][3]. First, we will check that the SUM-rank of U_n is also not full: $\mathrm{rk}_+ U_n \prec 1.89^n$.

2 Upper bound for the SUM-rank

Further, $||x||$ denotes the weight of a boolean vector x. Recall that a *code* with distance d (abbreviated, a d-code) in $\mathbb{B}^n$ is a set X such that $||x-y|| \geq d$ for any distinct $x, y \in X$.

Lemma 1. *The cube $\mathbb{B}^n$ can be partitioned into $2n$ parts that are 3-codes.*

Proof. Let N be the nearest number of the form $2^m - 1$ to n from above. We use the standard partition of the cube $\mathbb{B}^N$ into balls of radius 1 centered at the points of the Hamming 3-code C_0. Denote by C_i the set obtained from C_0 by inverting ith coordinates of the points — this is also a 3-code. By construction,

$$\mathbb{B}^N = C_0 \sqcup C_1 \sqcup \ldots \sqcup C_N.$$

Now it is easy to obtain the partition

$$\mathbb{B}^n = \Gamma_0 \sqcup \Gamma_1 \sqcup \ldots \sqcup \Gamma_N, \qquad \Gamma_i = \{c \in \mathbb{B}^n \mid (c, \vec{0}) \in C_i\}. \tag{1}$$

It remains to note that $N \leq 2n - 1$. ☐

Recall a simple relation for binomial coefficients:

$$\frac{1}{n} \cdot 2^{nH(k/n)} \preceq C_n^k \preceq 2^{nH(k/n)}, \qquad \text{as } \min\{k,\, n-k\} \to \infty, \tag{2}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the *binary entropy function* defined on the interval $[0,\, 1]$; at the endpoints, $H(0) = H(1) = 0$ by continuity.

Theorem 1. $\mathrm{rk}_+ U_n \preceq n^3 \left(\frac{3}{\sqrt[3]{4}}\right)^n \prec 1.89^n$.

Proof. Partition matrix U_n into submatrices $U_n^{p,q}$ with fixed row weights p and columns weights q. We restrict ourselves to the case $p + q \leq n$ (otherwise, the submatrix contains no ones). For each matrix $U_n^{p,q}$, we construct a partition into rectangles in one of two ways.

[3]The solution to the latter problem involves estimating the nonnegative real rank of a somewhat similar matrix. An implicit analogy with the bounding the rank of the unique disjointness matrix is explained in [9].

In the first way, apply a trivial partition by rows or by columns, depending on whether p or q is farther from $n/2$. The cardinality of such a partition does not exceed

$$\min\{C_n^p,\, C_n^q\} \le C_n^{(p+q)/2}.$$

In the second way, a matrix $U_n^{p,q}$ is first partitioned into submatrices U^S, whose rows and columns belong to (all possible) sets S of cardinality $p+q$. To construct a partition of a matrix U^S, apply partition (1) of the boolean cube $\mathbb{B}^{p+q}$ from Lemma 1. From any code set Γ_i, select the subset R_i of all vectors of weight p. Then $R_i \times \overline{R_i}$ defines the desired partition rectangle, where $\overline{X}$ denotes the elementwise inverse of the set X. The fact that the rectangle is legal is ensured by the property of R_i to be a 3-code. Indeed, since $||x-y|| \ge 3$ for distinct $x, y \in R_i$, vectors x and $\overline{y}$ have at least two common ones. The cardinality of the proposed partition of the matrix $U_n^{p,q}$ does not exceed $2nC_n^{p+q}$.

By (2), we obtain

$$\mathrm{rk}_+\, U_n^{p,q} \le \min\{C_n^{(p+q)/2},\, 2nC_n^{p+q}\} \preceq nC_n^{n/3} \le n2^{nH(1/3)} = n\left(\frac{3}{\sqrt[3]{4}}\right)^n,$$

since $H(1/3) = \log_2 3 - 2/3$.

It remains to sum the ranks of all $O(n^2)$ submatrices $U_n^{p,q}$. $\square$

3 Upper bound for the OR-Rank

A technical ingredient for the proof is the Sapozhenko—Lovasz—Stein lemma on the rank of a gradient covering [8, 10], see also [5, §2.6]. In a convenient and slightly weakened form, it is stated as follows.

Lemma 2. *Let $\mathcal{F} \subset 2^X$. If the elements of X are uniformly covered by sets from the family $\mathcal{F}$, then $\mathcal{F}$ contains a covering of X of size $w \le (1+\ln|X|)|X|/s$, where s is the average cardinality of a set in $\mathcal{F}$.*

Further, $X^{(k)}$ denotes the family of all subsets of cardinality k in X.

Theorem 2. $\mathrm{rk}_\vee\, U_n \le n^{O(1)}(3/2)^n$.

Proof. Consider again the partition of matrix U_n into submatrices $U_n^{p,q}$ with row weights p and column weights q. We will construct a covering for each matrix $U_n^{p,q}$ independently. Without loss of generality (the argument is symmetric), assume $p \ge q$. Denote $\alpha = p/n$ and $\beta = q/n$.

I. Degenerate case: $\beta \le 0.1$ or $\alpha \ge 0.9$.

Even the trivial partition of a matrix $U_n^{p,q}$ by rows or by columns has rank

$$C_n^{\min\{q,n-p\}} \preceq 2^{nH(0.1)} \prec 1.4^n.$$

In what follows, we assume that $\alpha, \beta \in [0.1, 0.9]$.

II. Main case: $\alpha \le 1/2$ and $\alpha + \beta \ge 1/2$.

We define a family of rectangles in $U_n^{p,q}$. It is obtained by combining the method of Lemma 1 and the standard technique of separation of variables[4]. Set $x = \alpha + \beta - 1/2$. Randomly partition $[n]$ into three groups: X_0 of $2xn$ elements, X_1 of $2(p - xn)$ elements, and X_2 of $2(q - xn)$ elements[5]. Applying (1), we obtain a partition of the family $X_0^{(xn)}$ into 3-codes:

$$X_0^{(xn)} = \mathcal{Y}_1 \sqcup \mathcal{Y}_2 \sqcup \ldots \sqcup \mathcal{Y}_{4xn}.$$

The row indices of a rectangle $R_{X_1,X_2,i}$ are composed of all possible combinations $a_0 \cup a_1$, where $a_0 \in \mathcal{Y}_i$ and $a_1 \in X_1^{(p-xn)}$, and the column indices are composed of combinations $b_0 \cup b_2$, where $b_0 \in X_0 \setminus \mathcal{Y}_i$ (the set difference operation is applied to each set in $\mathcal{Y}_i$) and $b_2 \in X_2^{(q-xn)}$.

The family $\mathcal{F} = \{R_{X_1,X_2,i}\}$ covers the ones of $U_n^{p,q}$ uniformly[6]. The average weight of a rectangle is

$$n^{\pm O(1)} C_{2xn}^{xn} C_{2(p-xn)}^{p-xn} C_{2(q-xn)}^{q-xn} = n^{-O(1)} 2^n.$$

Therefore, according to Lemma 2, $\mathcal{F}$ contains a covering of size

$$n^{O(1)} |U_n^{p,q}|/2^n \le n^{O(1)} \left(3/2\right)^n.$$

III. The case $\alpha + \beta \le 1/2$.

To construct a covering family, we use only separation of variables. Denote $\gamma = \frac{1}{\alpha+\beta}$. Randomly partition $[n]$ into two parts: X_1 of γp elements and X_2 of γq elements. The rows and columns of a rectangle R_{X_1} are numbered by sets from $X_1^{(p)}$ and $X_2^{(q)}$, respectively. The weight of any rectangle is

$$n^{\pm O(1)} C_{\gamma p}^{p} C_{\gamma q}^{q} = n^{\pm O(1)} 2^{nH(\alpha+\beta)}.$$

The family $\mathcal{F} = \{R_{X_1}\}$ covers the ones uniformly, hence it contains a covering of size

$$n^{\pm O(1)} |U_n^{p,q}|/2^{nH(\alpha+\beta)} = n^{\pm O(1)} C_n^{p+q} C_{p+q}^{p}/2^{nH(\alpha+\beta)} = n^{\pm O(1)} C_{p+q}^{p} \le n^{O(1)} \sqrt{2}^{\,n}.$$

[4]The latter approach allows to construct optimal coverings of matrices S_n, see [3].

[5]From here on, we neglect parity and roundings. Taking these factors into account only results in a refinement of the $n^{O(1)}$ factor in the estimates.

[6]The family is invariant under permutations in $[n]$.

IV. The case $\alpha \geq 1/2$.

To construct the family, we employ a combination of the two methods of Theorem 1. Let $z = \frac{\alpha\beta}{1-\alpha}$. Randomly choose a subset X of $q + zn$ elements in $[n]$. Consider a partition of the family $X^{(q)}$ into 3-codes:

$$X^{(q)} = \mathcal{Y}_1^X \sqcup \mathcal{Y}_2^X \sqcup \ldots \sqcup \mathcal{Y}_{2(q+zn)}^X.$$

A rectangle $R_{X,i}$ is formed by the columns $\mathcal{Y}_i^X$, and its row indices are all possible sets $a_0 \cup a_1$ of cardinality p, where $a_0 \in X \setminus \mathcal{Y}_i^X$ (the set difference operation is applied to every set in $\mathcal{Y}_i^X$) and $a_1 \subset [n] \setminus X$.

The proposed family $\mathcal{F} = \{R_{X,i}\}$ uniformly covers the ones of $U_n^{p,q}$. The average weight of a rectangle is

$$n^{\pm O(1)} C_{q+zn}^{zn} C_{n-q-zn}^{p-zn} = n^{\pm O(1)} 2^{nH(\alpha)},$$

since $\frac{zn}{q+zn} = \frac{p-zn}{n-q-zn} = \alpha$ due to the choice of z. Lemma 2 guarantees in $\mathcal{F} = \{R_{X,i}\}$ a covering for $U_n^{p,q}$ of size

$$n^{\pm O(1)} |U_n^{p,q}| / 2^{nH(\alpha)} = n^{\pm O(1)} C_n^p C_{n-p}^q / 2^{nH(\alpha)} = n^{\pm O(1)} C_{n-p}^q \leq n^{O(1)} \sqrt{2}^n.$$

Summing up the ranks of all submatrices yields the final estimate. □

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