# Complexity of circuit-type computations (survey)

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2026

# Evolution of the theory

1955–1975: classical results, construction of the theory's skeleton, formulation of the main problems

- theory of asymptotic synthesis of boolean functions, FFT algorithms, fast integer multiplication

1975–1995: development of the theory in depth, creation of more sophisticated methods in the main areas

- lower bounds for the complexity of monotone circuits, tensor methods of matrix multiplication, switching lemma

1995–2015: development of the theory in breadth, new models and directions (including quantum computing), active integration with related fields

- extended hierarchy of complexity classes, theory of communication complexity

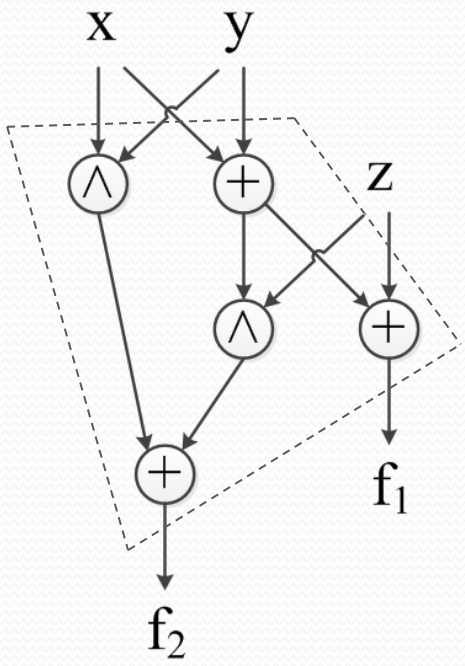
2015–present: in-depth study of peripheral domains, specification of combinatorial and probabilistic tools

- lower bound "lifting" methods, strengthened versions of the sunflower lemma

# Circuits (from f.e.): boolean, arithmetic

basis B

circuit S:




depth
$\mathbf{D}_B(S) = 3$

complexity
$\mathbf{C}_B(S) = 5$

$x + y + z = 2f_2 + f_1$

boolean bases: $B_2 = \{g(x,y)\}$, $B = \{\wedge, \oplus, 1\}$, ...

arithmetic bases: $A = \{+, \times, a \cdot x \mid a \in R\}$, ...

# Boolean circuits (lower bounds)

$\boxed{\mathsf{C}(f_n) \geq ?}$ $f_n$ — explicit boolean function of $n$ variables

for almost all functions: $\mathsf{C}(f_n) \asymp 2^n/n$ [Muller, 1956; Lupanov, 1958 (asymp.)]

method of constant substitutions — since 1950s:

over basis $\mathcal{B}_2 = \{g(x, y)\}$: $\mathsf{C}_{\mathcal{B}_2}(f_n) \gtrsim 3n$ [Blum, 1984]

over basis $\mathcal{U}_2 = \mathcal{B}_2 \setminus \{\oplus, \sim\}$: $\mathsf{C}_{\mathcal{U}_2}(f_n) \gtrsim 5n$ [Iwama, Morizumi, 2002]

over basis $\{\wedge, \neg\}$: $\mathsf{C}_{\{\wedge,\neg\}}(f_n) \gtrsim 7n$ [Red'kin, 1973]

method of affine substitutions [Find, Golovnev, Hirsch, Kulikov, 2015] $\rightarrow$ $\mathsf{C}_{\mathcal{B}_2}(f_n) \gtrsim 3.1n$ [Li, Yang, 2021]

$\boxed{\mathsf{C}(F_n) \geq ?}$ $F_n$ — boolean $(n, n)$-operator

$\mathsf{C}_{\mathcal{B}_2}(F_n) \gtrsim \mathsf{C}_{\mathcal{B}_2}(f_{n-o(n)}) + 2n \gtrsim 5.1n$ [Sergeev, 2026]

*Open problem:* $\mathsf{C}(f_n) = \omega(n)$ for some sequence $f_n$.

# Arithmetic circuits (lower bounds)

$\boxed{\mathsf{C}_{\mathcal{A}}(f_n) \geq ?}$ $\quad f_n \in \mathbb{R}[x],\ \deg f_n = n; \qquad \mathcal{A} = \{+, *\} \cup \mathbb{R}$ (or $\mathbb{C}$)

complexity of typical polynomials:

$\sim n$ additions, $\quad \sim n/2$ multiplications, $\quad \sim \sqrt{n} \ldots \sqrt{2n}$ nonscalar operations

[Motzkin; Pan; Belaga, 1950s; Paterson, Stockmeyer, 1973]

Lower bounds (based on applications of algebraic geometry):

[Strassen, 1974] $\ldots$ [Schnorr, Van de Wiele, 1978–80] [Heintz, Sieveking, 1980]

$\rightarrow$ $\sim n$ additions, $\quad \sim n/2$ multiplications, $\quad \Omega(\sqrt{n})$ nonscalar operations

Examples: $\quad \sum_{k=0}^{n} 2^{2^{kn^2}} x^k, \qquad \sum_{k=0}^{n} e^{2\pi \mathbf{i}/p_k} x^k, \quad \mathbb{P} = \{p_0, p_1, \ldots\}$

*Open problem:* construct explicit examples of hard-to-compute polynomials with $0/1$-coefficients.

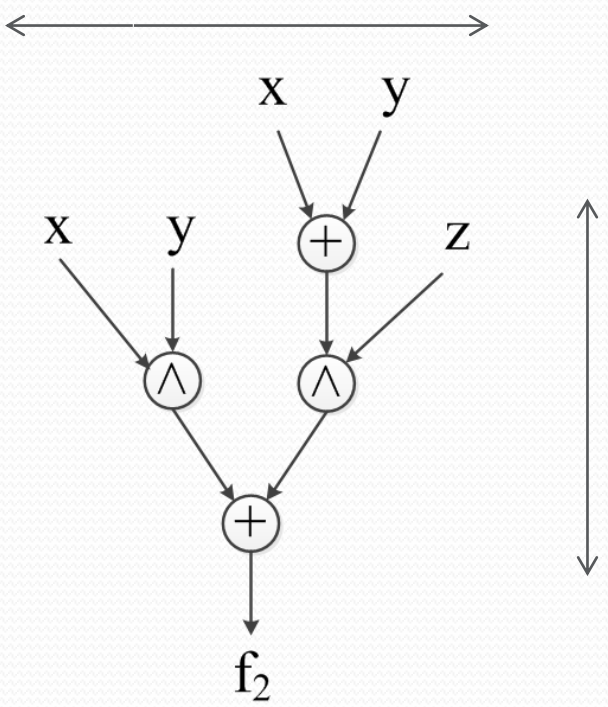
▷ for $f_n \in \mathbb{R}[x_1, \ldots, x_n]$, $\deg f = d$: $\quad \mathsf{C}_{\mathcal{A}}(f_n) \succeq n \log d$ [Baur, Strassen, 1983] $0/1$

Example: $\quad \sum_{k=1}^{n} x_i^d$

# Formulae

formula = circuit with gate fanout 1

$F = xy \oplus z(x \oplus y)$




complexity $\mathbf{\Phi}_B(F) = 5$

depth $\mathbf{D}_B(F) = 3$

# Boolean formulae

$\boxed{\Phi(f_n) \geq ?}$ $\quad f_n$ — explicit boolean function of $n$ variables

for a.a. functions: $\Phi(f_n) \asymp 2^n / \log n$ [Riordan, Shannon, 1942; Lupanov, 1960 (as.)]

over basis $\mathcal{B}_2 = \{g(x, y)\}$:

for universal functions: $\Phi_{\mathcal{B}_2}(f_n) \succeq \frac{n^2}{\log n}$ [Nechiporuk, 1966]

for symmetric functions: $\Phi_{\mathcal{B}_2}(f_n) \succeq n \log n$, $\Phi(f_n) \succeq n \log n$

[Fischer, Meyer, Paterson, 1982]; [Cherukhin, 2000]

over basis $\mathcal{B}_0 = \{\wedge, \vee, \neg\}$:

method of random substitutions $\rightarrow$ bounds for shrinkage exponents of bases

for Andreev function: $\Phi_{\mathcal{B}_0}(f_n) \succeq \frac{n^3}{\gamma(\log n)}$

$\gamma(x) = x^{7/2} \log^3 x$ [Håstad, 1998], $\gamma(x) = x^2 \log x$ [Tal, 2014]

for another function: $\Phi_{\mathcal{B}_0}(f_n) \succeq \frac{n^3}{\gamma(\log n)}$, $\gamma(x) = x \log^2 x$ [Tal, 2017]

*Open problem:* $\mathsf{C}_{\mathcal{B}_2}(Sym_n) \asymp n^{1+\Theta(1)}$, where $Sym_n$ — symmetric functions of $n$ variables?

# Depth vs complexity of formulae

*Open problem:* depth lower bound techniques independent of formula complexity lower bounds in complete arithmetic-type bases.

▷ in binary bases trivially holds: $\mathsf{D}(f) \geq \log_2 \Phi(f)$

$\boxed{\mathsf{D}_{\mathcal{B}}(f) \asymp \log \Phi_{\mathcal{B}}(f)}$ in boolean and arithmetic bases $\mathcal{B}$

[Khrapchenko, 1968] [Brent, Kuck, Maruyama, 1973] [Ugol'nikov; Ragaz, 1987]

Examples:

$\mathcal{B}$ — nonarithm. complete: $D_{\mathcal{B}}(f_n) \gtrsim 2\log_2 \Phi_{\mathcal{B}}(f_n)$, $\mathcal{B} = \{\backslash\}$ [Khrapchenko, 1967]

mon. arithm. $\mathcal{A}_+ = \{+, *\}$: $D_{\mathcal{A}_+}(f_n) \gtrsim 1.5\log_2 \Phi_{\mathcal{A}_+}(f_n)$ [Coppersmith, Schieber, 1992]

mon. boolean $\mathcal{B}_M = \{\vee, \wedge\}$: $D_{\mathcal{B}_M}(f_n) \gtrsim 1.06\log_2 \Phi_{\mathcal{B}_M}(f_n)$ [Sergeev, 2019]

Upper bounds:

for a.a. functions: $\mathsf{D}(f_n) \asymp n$ [Riordan, Shannon, 1942; Lupanov, 1970 (asymp.)]

*Open problem:* can we prove $\mathsf{D}(\mathrm{GCD}_n) \ll n$, where $\mathrm{GCD}_n$ is GCD operator of $n$-bit integers?

▷ $\mathsf{C}(\mathrm{GCD}_n) \preceq n\log^2 n$; polynomial GCD may be implemented by parallel circuits

# Monotone boolean circuits (lower bounds)

$\boxed{\mathsf{C}_{\mathcal{B}_M}(f_n) \geq ?}$ $f_n$ — monotone boolean function of $n$ variables; $\mathcal{B}_M = \{\vee, \wedge\}$

for a.a. functions: $\mathsf{C}_{\mathcal{B}_M}(f_n) \asymp 2^n/n^{3/2}$ [Red'kin, 1979; Andreev, 1988 (asymp.)]

method of DNF/CNF approximations [Andreev; Razborov, 1985] $\rightarrow$

$\mathsf{C}_{\mathcal{B}_M}(f_n) \succeq 2^{n^{1/3-o(1)}}$ [Andreev, 1987]

$\mathsf{C}_{\mathcal{B}_M}(Per_n) \succeq n^{\Omega(\log n)}$, $Per_n$ — logical permanent [Razborov, 1985]

$\mathsf{C}_{\mathcal{B}_M}(f_n) \succeq 2^{n^{1/6-o(1)}}$, $\mathsf{C}(f_n) = n^{O(1)}$ [Tardos, 1987]

appropriate modification of the Sunflower lemma $\rightarrow$

$\mathsf{C}_{\mathcal{B}_M}(f_n) \succeq 2^{n^{1/2-o(1)}}$ [Cavalar, Kumar, Rossman, 2020]

$\mathsf{C}_{\mathcal{B}_M}(Per_n) \succeq 2^{n^{1/3-o(1)}}$ [Cavalar, Göös, Riazanov, Sofronova, Sokolov, 2025]

+ lifting lower bounds method $\rightarrow$

$\mathsf{C}_{\mathcal{B}_M}(f_n) \succeq 2^{n^{1/3-o(1)}}$, $\mathsf{C}(f_n) = n^{O(1)}$ [de Rezende, Vinyals, 2025]

*Open problem:* $\mathsf{C}_{\mathcal{B}_M}(Maj_n) \succ n^2$, where $Maj_n$ is the majority function?

# Monotone arithmetic circuits (lower bounds)

$\boxed{\mathsf{C}_{\mathcal{A}_+}(f_n) \geq ?}$ $f_n \in \mathbb{R}_+[x_1, \dots x_n]$ — multilinear polynomial;

$\boxed{\mathsf{C}_{\mathcal{A}_+}(g_n) \geq ?}$ $g_n \in \mathbb{R}_+[x]$, $\deg g_n = n$; $\mathcal{A}_+ = \{+, *\} \cup \mathbb{R}_+$

for random polynomials: $\mathsf{C}_{\mathcal{A}_+}(f_n) \asymp 2^n$, $\mathsf{C}_{\mathcal{A}_+}(g_n) \asymp n$.

a set of monomials doesn't contain computationally convenient structures $\rightarrow$

$\mathsf{C}_{\mathcal{A}_+}(f_n) \succeq 2^{n^{1/2-o(1)}}$ [Schnorr, 1976], [Jerrum, Snir, 1982] $0/1$-coeff.

$\dots$ $\mathsf{C}_{\mathcal{A}_+}(f_n) \succeq 2^{n-o(n)}$, $\mathsf{C}_{\mathcal{A}_+}(g_n) \succeq n^{1-o(1)}$ [Gashkov, Sergeev, 2012] $0/1$

$\mathsf{C}_{\mathcal{A}_+}(f_n) = 2^{\Omega(\sqrt{n})}$, $\mathsf{C}_{\mathcal{A}}(f_n) = n^{O(1)}$ [Valiant, 1980] $0/1$

$\mathsf{C}_{\mathcal{A}_+}(f_n) = 2^{\Omega(n)}$, $\mathsf{C}_{\mathcal{A}}(f_n) = n^{O(1)}$ coeff.$\in n^{O(1)}$

[Cavalar, Fabris, Mukhopadhyay, Srinivasan, Yehudayoff, 2025]

*Open problem:* example of multilinear polynomial $\mathsf{C}_{\mathcal{A}_+}(f_n) = 2^{\Omega(n)}, \mathsf{C}_{\mathcal{A}}(f_n) = n^{O(1)}$ with $0/1$-coefficients, and of single-variable: $\mathsf{C}_{\mathcal{A}_+}(g_n) = n^{\Omega(1)}, \mathsf{C}_{\mathcal{A}}(g_n) = \log^{O(1)} n$.

# Monotone arithmetic circuits

Effect of division $\quad \mathcal{A}_{/} = \{+, *, /\} \cup \mathbb{R}, \qquad \mathcal{A}_{+,/} = \{+, *, /\} \cup \mathbb{R}_{+}$

$$\mathsf{C}_{\mathcal{A}}(f) \preceq d \log d \cdot \mathsf{C}_{\mathcal{A}_{/}}(f), \quad f \in \mathbb{R}[X], \quad d = \deg f \qquad \text{[Strassen, 1973]}$$

(!) in $\mathcal{A}_{+,/}$ one can compute nonmon. polynomials: $x^2 - xy + y^2 = (x^3 + y^3)/(x + y)$

$$\mathsf{C}_{\mathcal{A}_{+,/}}(ST_n) \preceq n^3, \quad \mathsf{C}_{\mathcal{A}_{+}}(ST_n) \succeq 2^{\Omega(\sqrt{n})} \qquad ST_n - \text{Kirchhoff polynomial}$$

(enumerating spanning trees in the complete $n$-vertex graph)

[Fomin, Grigoriev, Koshevoy, 2016] [Jerrum, Snir, 1982; Jukna, Seiwert, 2019]

Tropical circuits (dynamic programming algorithms): $\mathcal{T}_{+} = \{\min, +\} \cup \mathbb{R}_{+}$

*Open problem:* is it true that $\mathsf{C}_{\mathcal{T}_{+}}(PATH_n) \asymp n^3$, where $PATH_n$ is the tropical polynomial computing the shortest path between two poles in an $n$-vertex graph (optimality of the Bellman–Ford–Moore algorithm)?

▷ computational complexity of shortest paths between all pairs is $\Theta(n^3)$

# Depth and complexity

class $\mathsf{NC}^k$: (uniform) circuits of complexity $n^{O(1)}$ and depth $O(\log^k n)$

class $\mathsf{NC}^k_{\mathcal{A}}$: the same for arithmetic setting

multiplication, division $\in \mathsf{NC}^1$ [folclore; Beame, Cook, Hoover, 1986]

«$\mathsf{P}_{\mathcal{A}} = \mathsf{NC}^2_{\mathcal{A}}$» for arithm. circuits [Valiant, Skyum, Berkowitz, Rackoff, 1983]

in particukar, determinant $\in \mathsf{NC}^2_{\mathcal{A}}$
[Berkowitz, 1982; Borodin, von zur Gathen, Hopcroft, 1982]

polynomial GCD $\in \mathsf{NC}^1_{\mathcal{A}}$ (piecewise) [Andrews, Wigderson, 2024] $\mathbb{R}$, $\mathbb{C}$

*Open problem:* logical permanent $\in \mathsf{NC}^k$?

▷ the existence of circuits of complexity $n^{O(1)}$ and depth $O(\log^2 n)$ is proven [BGH82]

*Open problem:* can the operator of maximum of $n$ numbers of length $n$ be implemented by boolean circuits of complexity $O(n^2)$ and depth $O(\log n)$?

# Multiplication

$\boxed{\mathsf{C}(M_n) \geq ?}$ $\quad M_n$ — operator of multiplication of $n$-bit integers

$\mathsf{C}(M_n) \preceq n^{\log_2 3}$ [Karatsuba, 1960] $\quad \rightarrow$

$+$ interpolation [Toom], $\quad +$ FFT [Bakhvalov, Karatsuba]

$\rightarrow \quad \mathsf{C}(M_n) \preceq n \log n \log\log n$ [Schönhage, Strassen, 1971]

$+$ special choice of interpolation rings $\rightarrow$

$\mathsf{C}(M_n) \preceq n \log n \cdot 2^{O(\log^* n)}$ [Fürer, 2007]

$+$ approximation of $\mathrm{FFT}_N$ by $\mathrm{FFT}_{2^n}$ $\rightarrow$

$\mathsf{C}(M_n) \preceq n \log n$ [Harvey, van der Hoeven, 2019]

$\boxed{\mathsf{C}_{\mathcal{A}}(M_n^R) \geq ?}$ $M_n^R$ — operator of multiplication of polynomials deg $< n$ from $R[x]$

$\mathsf{C}_{\mathcal{A}}(M_n^{\mathbb{C}}) \preceq n \log n$ [Cooley, Tukey, 1965] $\quad$ FFT

$\mathsf{C}_{\mathcal{A}}(M_n^R) \preceq n \log n \log\log n$ [Cantor, Kaltofen, 1991] $\quad$ arbitrary ring $R$

$\mathsf{C}_{\mathcal{A}}(M_n^R) \preceq n \log n \cdot 2^{O(\log^* n)}$ [Harvey, van der Hoeven, Lecerf, 2014]

*Open problem:* $\mathsf{C}_{\mathcal{A}}(M_n^R) \preceq n \log n$?

▷ holds for $\mathbb{R}, \mathbb{C}$; $\quad$ there is a conventional algorithm [HvdH19]

# Matrix multiplication

$\boxed{\mathsf{C}_{\mathcal{A}}(MM_n) \geq ?}$ $MM_n$ — operator of multiplication of $n \times n$-matrices (over $R$)

$\mathsf{C}_{\mathcal{A}}(MM_n) \preceq n^{\log_2 7}$ [Strassen, 1968]

1970–1980s:

— trilinear representation [Pan]

— approximate computations (border rank) [Bini]

— method of direct sums [Schönhage]

— laser method [Strassen]

$\ldots$ $\mathsf{C}_{\mathcal{A}}(MM_n) \preceq n^{2.376}$ [Coppersmith, Winograd, 1986]

analysis of powers of Coppersmith–Winograd tensor [Stothers, 2010]

$\ldots$ $\mathsf{C}_{\mathcal{A}}(MM_n) \preceq n^{2.372}$

[Alman, Duan, Vassilevska Williams, Xu, Xu, Zhou, 2024]

*Open problem:* $\mathsf{C}_{\mathcal{A}}(MM_n) = n^{2+o(1)}$?

Multiplication of $n \times n$- and $n \times n^\alpha$-matrices: $\mathsf{C}_{\mathcal{A}}(MM_{n,n,n^\alpha}) = n^{2+o(1)}$

$\alpha > 0.294$ [Coppersmith, 1996] $\ldots$ $\alpha > 0.321$ [WXXZ, 2023]

# Matrix multiplication, modular composition

Practically meaningful methods (economical trilinear decompositions):

$\mathsf{C}_{\mathcal{A}}(MM_n) \preceq n^{2.774}$ [Pan, 1982], [Schwartz, Zwecher, 2025]

Alternative representations: example for $MM_2$:

standard: 7 multiplications, 15 additions [Strassen, Winograd]

alternative: 7 multiplications, 12 additions [Bodrato, 2010]

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \to \begin{bmatrix} x_{11} & x_{12} - x_{21} + x_{22} \\ x_{22} - x_{21} & x_{12} + x_{22} \end{bmatrix}$$

Modular composition $\boxed{\mathsf{C}_{\mathcal{A}}(MC_n) =?, \quad \mathsf{C}_{\mathcal{A}}(MC_n^*) =?}$

$MC_n$ computes $f(g(x)) \bmod h(x)$; $MC_n^*$ computes $f(g(x)) \bmod x^n$, $\deg f, g, h = n$

$\mathsf{C}_{\mathcal{A}}(MC_n) \prec n^{1.7}$, $\mathsf{C}_{\mathcal{A}}(MC_n^*) \preceq \mathsf{C}_{\mathcal{A}}(M_n) \cdot \sqrt{n \log n} = n^{1.5+o(1)}$ [Brent, Kung, 1978]

interpolation idea $\to$ $\mathsf{C}_{\mathcal{A}}(MC_n) = n^{1+o(1)}$ [Umans, 2008] $GF(p^k)$, $p$ is small

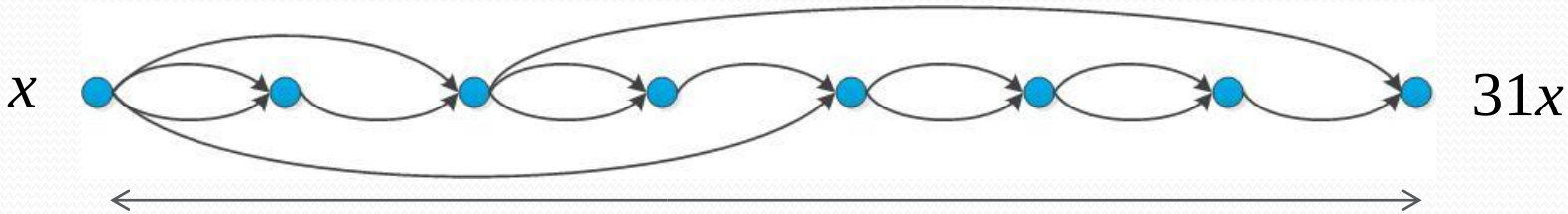
reduction to computing of a member of recurrent sequence $\to$

$\mathsf{C}_{\mathcal{A}}(MC_n^*) \preceq \mathsf{C}_{\mathcal{A}}(M_n) \cdot \log n$ [Kinoshita, Li, 2024]

*Open problem:* $\mathsf{C}_{\mathcal{A}}(MC_n) = n^{1+o(1)}$ in general case (at least for $\mathbb{R}$ и $\mathbb{C}$)?

# Addition chains

$C$: 1, 2, 3, 6, 7, 14, 28, 31




length $\mathbf{L}(C) = 7$

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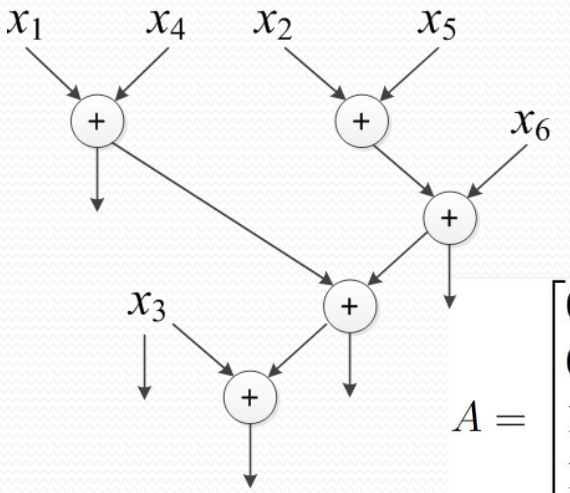
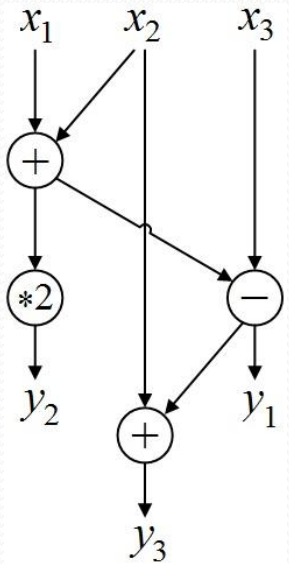
## Additive circuits




$$y = A \cdot x$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

## Linear (arithmetic) circuits




$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

# Linear circuits (lower bounds)

addition chains: $\mathsf{L}(n) \sim \log_2 n$ [Brauer, 1939]

additive circuits: $\mathsf{L}(A_n) \asymp n^2/\log_2 n$ for $0/1$ $n \times n$-matrices
[Lupanov, 1956; Nechiporuk, 1963 (asymp.)]

linear circuits: $\mathsf{L}(A_n) \asymp n^2$ for $n \times n$-matrices over $\mathbb{R}$ (or $\mathbb{C}$)

Lower bounds:

monot. comput. of $0/1$ matrices (over $(\mathbb{B}, \vee)$ or $\mathbb{Z}_+$): $\mathsf{L}(A_n) = n^{2-o(1)}$ [Andreev, 1986]

circuits over $\mathbb{C}$: $\mathsf{L}(A_n) \succeq n^2/\log n$ [Heintz, Sieveking, 1980]

*Open problem:* addition chains: construct an explicit set $\mathbf{m} = \{m_1, \ldots, m_n\}$, $m_i \in n^{O(1)}$, of complexity $\mathsf{L}(\mathbf{m}) = n + \Omega(n)$.

▷ upper bound $\mathsf{L}(\mathbf{m}) = n + O(n)$; known result: $\mathsf{L}(\mathbf{m}) = n + \Omega(n^{1-\varepsilon})$ [Sergeev, 2024]

*Open problem:* circuits over $GF(2)$: $\mathsf{L}(A_n) = \omega(n)$ for some sequence of matrices $A_n \in GF(2)^{n \times n}$.

▷ known bound: $\mathsf{L}(A_n) \gtrsim 5n$ [Sergeev, 2025]

*Open problem:* linear circuits over $\mathbb{R}$ (or $\mathbb{C}$): $\mathsf{L}(A_n) = \omega(n)$ for a sequence of $n \times n$-matrices $A_n$ with $0/1$-coefficients.

▷ there are bounds for circuits with bounded constants

# Communication complexity

A

B

… 0 1 0 … 1 1 0 1 … 0 1 …

game G[f]

knows X

knows Y

! have to determine f(X,Y) via as few communicated bits as possible

cc(G[f]) – minimal possible number of communicated bits

# Communication complexity and depth

$\boxed{D[f]}$ $\quad A: \ x \in f^{-1}(1) \qquad B: \ y \in f^{-1}(0) \qquad$ determine $i: \ x_i \neq y_i$.

$\boxed{D_+[f]}$ $\quad f$ — monot., $\quad A: \ x \in f^{-1}(1) \quad B: \ y \in f^{-1}(0) \quad$ determine $i: \ x_i > y_i$.

$$\mathsf{cc}(D[f]) = \mathsf{D}_{\mathcal{B}_0}(f), \quad \mathcal{B}_0 = \{\wedge, \vee, \neg\}$$
$$\mathsf{cc}(D_+[f]) = \mathsf{D}_{\mathcal{B}_M}(f), \quad \mathcal{B}_M = \{\wedge, \vee\} \quad \text{[Karchmer, Wigderson, 1990]}$$

Applications:

$\mathsf{D}_{\mathcal{B}_M}(Stcon_n) \asymp \log^2 n$, $\quad Stcon_n$ — $st$-connectivity function of an $n$-vertex graph
[Karchmer, Wigderson, 1990]

$\mathsf{D}_{\mathcal{B}_M}(Per_n) \asymp n$, $\quad Per_n$ — logical permanent $\quad$ [Raz, Wigderson, 1992]

$\mathsf{D}_{\mathcal{B}_M}(CL_{n,k}) \asymp k \log n \quad$ for $k \leq n^{\varepsilon}$, $\quad CL_{n,k}$ — $k$-clique function for a graph
[Raz, McKenzie, 1997]

$\boxed{CIS[G]}$ $\quad A$ : clique $a$ in a graph $G \quad B$ : independent set $b \subset G \qquad$ find $|a \cap b|$

for a graph $G = (V, E)$ one has $\mathsf{cc}(CIS[G]) \preceq \log^2 |V| \qquad$ [Yannakakis, 1991]

lifting lower bounds method [Göös, 2015] $\quad \rightarrow$

$$\exists G: \quad \mathsf{cc}(CIS[G]) \asymp \log^{2-o(1)} |V| \qquad \text{[Balodis, 2021]}$$

# topics beyond the scope of the presentation:

- Turing machines, programs with memory
- circuits over infinite bases, circuits of bounded depth
- decision trees, contact circuits, and branching programs
- nondeterministic circuits and other models
- quantum computing
- probabilistic (randomized) algorithms
- average complexity / approximate computations
- noncommutative computations
- combinatorial optimization problems
- number-theoretic algorithms
- classification problems of complexity class hierarchies