On the meaning of works by V. M. Khrapchenko*

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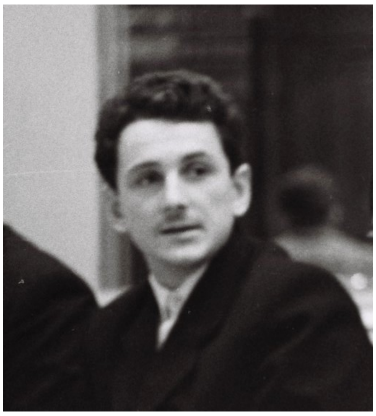

Abstract

The present paper surveys main works and results by Valery Mikhailovich Khrapchenko, who stands among the pioneers of national theoretical cybernetics.

Valery Mikhailovich Khrapchenko (1936–2019) was born in the family of the famous literary scholar, later academician of the USSR Academy of Sciences Mikhail Borisovich Khrapchenko and music teacher Tamara Erastovna Tsytovich. He graduated from the Moscow Power Engineering Institute, Faculty of Automation and Computer Engineering. From 1959 to 1966, he worked at the Institute of Electronic Control Machines and studied there in graduate school.

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The early works of V. M. Khrapchenko [1, 2], related to the subject of constructing computers and their algorithmic support, attracted the attention of Sergei Vsevolodovich Yablonskii and Oleg Borisovich Lupanov, the leaders of Soviet discrete mathematics and mathematical cybernetics. In 1966, Khrapchenko moved to their department of theoretical cybernetics at the Institute of Applied Mathematics, which later received the name of M. V. Keldysh. Valery Mikhailovich worked in this department for the rest of his life, over 50 years.

V. M. became especially close with O. B. Lupanov, whom he considered his scientific advisor. Under his influence, Khrapchenko's scientific interests shifted to the theoretical plane, in particular, they turned to the problem of lower complexity bounds. Nevertheless V. M. always found problems that had a direct connection with electronics: how to quickly add and multiply numbers, how to reduce the computation time in a circuit, etc. Khrapchenko, who knew Oleg Borisovich like no one else, recently wrote his biography [22], containing many little-known details.

V. M. Khrapchenko often gave reports at the weekly seminars "Mathematical Problems of Cybernetics" and "Synthesis of Control Systems" under the direction of S. V. Yablonskii and O. B. Lupanov at Moscow University. According to many participants, he was the most popular speaker. Valery Mikhailovich always carefully prepared for the seminars and explained the results so clearly that everyone understood.

Mathematical results of V. M. Khrapchenko can be attributed to five directions: synthesis of parallel adders [3, 13, 19], relations between complexity and depth of Boolean formulas [3, 4, 12, 14], lower bounds for complexity of formulas [5, 6, 8, 9, 18, 21], synthesis of formulas for symmetric Boolean functions [7, 9, 10], relations between depth and delay of circuits [11, 13, 16, 20].

Further, the functionals of the formula complexity and depth over a basis B are denoted by L_B and D_B, respectively. For definitions, see [43, 48, 31, 39, 60, 66].

Parallel adders.

One of the first practical problems of complexity theory was the synthesis of parallel adder circuits. Let us denote the Boolean operator of addition of n-digit binary numbers by Σ_n. In essence, the computation time of the adder Σ_n is determined by the method of computing the system of carries, which, when appropriate notations are introduced, takes the form

$$f_k = y_1 \vee y_2x_1 \vee y_3x_2x_1 \vee \ \dots \ \vee y_kx_{k-1} \cdot \ldots \cdot x_1, \qquad k = 1, \ldots, n. \quad (1)$$

The relation $D_B(\Sigma_n) \leq D_B(f_n)+O(1)$ holds in an arbitrary complete Boolean finite basis B. The bound $D_B(f_n) = O(\log n)$ is trivial to prove, but such an important basic operation as addition required more precise knowledge. In his first widely known paper [3] dated by 1967, Khrapchenko obtained an asymptotically tight result $D_{B_0}(\Sigma_n) \sim \log_2 n$ for the standard basis $B_0 = \{\vee, \wedge, \bar{\ }\}$, which follows from the accurate upper bound

$$D_{B_0}(f_n) \leq \log_2 n + \sqrt{2 \log_2 n} + O(1). \tag{2}$$

This bound is obtained recursively by applying formulas of the form

$$f_{kr} = f_r(\tilde{x}_1) \vee (x_1 \cdot \ldots \cdot x_r) f_r(\tilde{x}_2) \vee (x_1 \cdot \ldots \cdot x_{2r}) f_r(\tilde{x}_3) \vee \ldots \\ \ldots \vee (x_1 \cdot \ldots \cdot x_{(k-1)r}) f_r(\tilde{x}_k), \tag{3}$$

where $\tilde{x}_i$ are suitable groups of variables.

Moreover, V. M. showed that for any finite Boolean basis B, we have $D_B(\Sigma_n) \sim \tau_B \log_2 n$, where the constant τ_B can be determined from the asymptotic relation $D_B(x_1 \cdot \ldots \cdot x_n) \sim \tau_B \log_2 n$. In this case, the value of the constant is always within the range $0 < \tau_B \leq 2$. An example of a basis for which $\tau_B = 2$ is the basis B_S of the single function "Scheffer stroke" $x \mid y$.

Direct application of the formulas providing estimate (2) leads to an adder of nonlinear complexity. Employing a more flexible construction, at the cost of a small degradation in depth, Khrapchenko constructed an adder of depth $\log_2 n + O(\sqrt{\log n})$ and linear complexity $(12 + o(1))n$. Similarly, an adder of complexity $O(n)$ and depth $(\tau_B + o(1)) \log_2 n$ can be constructed in an arbitrary basis.

A little later, bound (2) was also derived by R. Brent [25], but the estimate he obtained for the complexity of an adder of depth $(1 + o(1)) \log_2 n$ was $O(n \log n)$. In [33] it is shown that the depth of an adder of linear complexity can be reduced to $\log_2 n + \sqrt{(2 + o(1)) \log_2 n}$ (in the basis B_0, the complexity estimate, as in [3], is $(12 + o(1))n$).

In 2008, M. I. Grinchuk [34] improved Khrapchenko's bound to $D_{B_0}(\Sigma_n) \leq \log_2 n + \log_2 \log n + O(1)$. In fact, he established the complexity of functions f_n in the monotone basis $B_M = \{\vee, \wedge\}$. Comparing his upper bound with the lower bound of B. Commentz-Walter [28], we obtain

$$D_{B_M}(f_n) = \log_2 n + \log_2 \log n \pm O(1).$$

By combining the methods of [33] and [34], the depth of a linear-complexity adder can be estimated as $\log_2 n + O(\log^2 \log n)$. Recently, in a continuation of this line of results, S. Held and S. Shpirkl [37] constructed adders of depth $\log_2 n + O(\sqrt{\log n})$ and linear complexity with an additional constraint of 2 on the fan-out of elements.

Other known constructions of parallel adders, such as prefix adders, have depth and complexity asymptotically greater than those of the parallel adders of Khrapchenko and Grinchuk, although they may have an advantage for small n.

From the standpoint of lower bounds, B. Commentz-Walter and J. Sattler [29] established

$$D_{B_0}(f_n) \geq \log_2 n + (1 - o(1)) \log_2 \log \log n. \tag{4}$$

Observing that $D_{B_0}(f_n) \leq D_{B_0}(\Sigma_n) + O(1)$, Khrapchenko in [19] deduced as a corollary that the lower bound (4) also holds for the depth of the adder $D_{B_0}(\Sigma_n)$. This result is the first and so far the only nontrivial lower bound for the depth of addition over the standard basis.

The formulas of Khrapchenko's method, on which the bound (1) is achieved, do not use the specifics of Boolean algebra and therefore can be applied to compute expressions of the form

$$y_1 + y_2 x_1 + y_3 x_2 x_1 + \ \ldots \ + y_n x_{n-1} \cdot \ldots \cdot x_1 \tag{5}$$

over the arithmetic basis $\{+, *\}$ in an arbitrary ring, see (3). In contrast to the formulas of Grinchuk's method. A special case of expressions (5) are polynomials of single variable: set $x_1 = \ldots = x_{n-1} = x$, and consider y_i as coefficients. Thus, Khrapchenko's circuit turns into a method for evaluating a polynomial at a point in $\log_2 n + \sqrt{2 \log_2 n} + O(1)$ parallel steps. In the early 1970s, K. Maruyama [44], I. Munro, and M. Paterson [46] rediscovered this method. Later, S. R. Kosarayu [41] proved that the computation depth in the problem under discussion cannot be less than $\log_2 n + \sqrt{(2 - o(1)) \log_2 n}$. Thus, in the general algebraic setting, Khrapchenko's method is optimal in a stronger sense than in the Boolean domain.

Formula depth and complexity.

It is more convenient to study computation depth in the formula model, since the depth of a function when implemented by circuits and formulas over the same basis coincides, however, the formula is a simpler object (in graphical representation, is a tree). The relation $D_B(f) \geq \log_k L_B(f)$ is trivially satisfied if the basis B consists of no more than k-input functions. But it was Khrapchenko who first noted in 1967 that the inequality also holds in the other direction:

$$D_B(f) = O(\log L_B(f)) \tag{6}$$

is valid for any Boolean complete finite basis B. Unfortunately, Khrapchenko's result was reflected only in one paragraph of the report [4]. Therefore, priority is often given to P. Spira, who proved (6) for the basis B_0 in [62].

Bases satisfying (6) are usually called uniform. Following the work of Khrapchenko [14], to characterize the uniformity of a basis B it is convenient to introduce a quantity (taking constant values or ∞)

$$c_B = \varlimsup_{N\to\infty} \max_{L_B(f)=N} \frac{D_B(f)}{\log_2 N}.$$

The definition means that for any function f expressed in a basis B, we have $D_B(f) \le (c_B + o(1)) \log_2 L_B(f)$, and there exists an infinite sequence of functions f_k for which $D_B(f_k) \ge (c_B - o(1)) \log_2 L_B(f_k)$.

In new terms, the result [4] can be formulated as $c_B < \infty$ for Boolean finite complete bases. In fact, the argument works for any complete finite bases, not necessarily Boolean. Proving $c_B < \infty$ for incomplete Boolean finite bases turned out to be a difficult problem. 20 years passed before A. B. Ugol'nikov [65] and M. Ragas [56] independently solved it. They also constructed examples of non-uniform bases ($c_B = \infty$) in three-valued logic.

The pioneering works [4, 62] were followed by a series of results dealt with bounds on the uniformity constants of various bases, mainly Boolean or arithmetic. Curiously, the first nontrivial lower bound was actually obtained by Khrapchenko in an earlier paper [3] for the basis $B_S = \{x \mid y\}$. From the lower bound $D_{B_S}(x_1 \cdot \ldots \cdot x_n) \ge 2\log_2 n$ proved in [3] and the trivial upper bound $L_{B_S}(x_1 \cdot \ldots \cdot x_n) = O(n)$ it follows that $c_{B_S} \ge 2$. Later, W. McColl [45] rediscovered this result and obtained several upper and lower bounds for the uniformity constants of other Boolean bases.

In [12] Khrapchenko obtained currently record bound $c_{B_0} < 1.73$ for the uniformity constant of the standard basis, strengthening the result [53]. The proof was carried out via the usual method for such problems of parallel reconstructing of formulas, but with a deeper case enumeration.

A little later in [14] Khrapchenko considered another rather popular in Boolean complexity theory basis $B_3 = \{xy \vee xz \vee yz, {}^-, 0, 1\}$. For the uniformity constant of this basis, he managed to obtain close estimates $1 \le c_{B_3} < 1.45$. Note that the same ratio between the estimates of the uniformity constant is also known for the basis B_S: $2 \le c_{B_S} < 2.89$ (the upper bound is proved in [45]). Then the estimates for the monotone arithmetic basis $B_A = \{+, *\}$ were brought even closer together: $1.5 \le c_{B_A} \le 2$ (the lower bound was proved in [30], and the upper one in [41]).

It should be noted that among the results on the uniformity constants of usual Boolean or arithmetic bases there are only a few nontrivial lower bounds, two of which are due to Khrapchenko. Exact constants have not yet been established for any of the "interesting" bases. In particular, the problem of obtaining nontrivial lower bounds for the uniformity constants of sufficiently expressive complete bases such as B_0 or the arithmetic basis $\{+, -, *\}$ has not been solved.

Formula lower complexity bounds.

Around 1970, Khrapchenko obtained his most famous result, a method for establishing lower complexity bounds of formulas in the standard basis, which is described by the relation

$$L_{B_0}(f) \geq \frac{|R(N_0, N_1)|^2}{|N_0| \cdot |N_1|}, \tag{7}$$

where N_0 and N_1 are arbitrary sets of zeros and ones, respectively, of the function f, and $R(A, B)$ is the set of pairs $(\alpha \in A, \beta \in B)$ of Boolean vectors that differ in exactly one position.

The method gives the highest possible bound $L_{B_0}(l_n) \geq n^2$ for linear functions $l_n = x_1 \oplus \ldots \oplus x_n \oplus \sigma$, where $\sigma \in \mathbb{B} = \{0, 1\}$. Initially, Khrapchenko obtained a new lower bound for the complexity of a linear function [5], and then turned the proof into a general method [6][1]. Strictly speaking, the bound was proved in the model of parallel-sequential switching circuits (π-circuits). But in essence, a π-circuit is an alternative way of representing a formula.

The result [5] provides a practically final solution for the problem of complexity of a linear function, since a simple method of synthesis [67] for an arbitrary n leads to $L_{B_0}(l_n) < (9/8)n^2$, and for $n = 2^k$ simply to $L_{B_0}(l_n) \leq n^2$. Thus, this problem of the complexity of a linear function, known as the S. V. Yablonskii problem [67], after Khrapchenko's work was reduced to closing the remaining small gap between the lower and upper bounds.

Before the appearance of Khrapchenko's method, the complexity of a linear function was estimated as $\Omega(n^{3/2})$ by B. A. Subbotovskaya's method [63], and the maximum known lower bounds for the complexity of specific functions were of the order of $n^2/\log n$ (E. I. Nechiporuk's method [47]; however, this method works for any complete basis). Subsequently, A. E. Andreev [24] proposed a generalization of Subbotovskaya's method — a method for compressing formulas under random restrictions on inputs — and for the complexity of a specially devised function he obtained a lower bound of the form $n^{2.5-o(1)}$. But before Andreev's work, the bounds of Khrapchenko's method remained record-breaking.

As a consequence of (7) Khrapchenko established [6] that the complexity of almost all symmetric functions is of order at least n^2, in particular, quadratic lower bounds hold for the majority function and many threshold

[1] In [21] Khrapchenko proposed a shortened proof of (7) — approximately as it is usually presented in lecture courses at Moscow University.

functions, for elementary symmetric functions, in general for any functions taking different values on neighboring layers in the middle part of the Boolean cube, and also for the determinant of a Boolean matrix. In [8] Khrapchenko showed that (7) implies quadratic lower bounds for binary components of real functions with a continuous and not identically zero second derivative. In particular, the binary digits of the product, quotient, and many analytic functions have at least quadratic complexity. In [55] A. K. Pulatov's applied Khrapchenko's method to derive lower bounds for the complexity of characteristic functions of some codes.

The method was further developed. K. L. Rychkov [57] proposed an interpretation of Khrapchenko's method in terms of coverings of bipartite graphs (see also [39]) and obtained the following generalization of (7):

$$L_{B_0}(f) \geq \frac{|R_t(N_0, N_1)|^2}{\left(1 + C_{n-1}^1 + \ldots + C_{n-1}^t\right) |N_0| \cdot |N_1|}, \tag{8}$$

where n is the number of arguments of f, and either the set N_0 (the zeros of the function) or N_1 (the ones of the function) have pairwise distances (between their elements) no less than $2t+1$, and $R_t(A, B)$ is the set of pairs $(\alpha \in A, \beta \in B)$ of vectors with distance at most $t+1$. The indicated modification of the method made it possible to obtain high, up to quadratic, lower bounds for the characteristic functions of arbitrary dense codes. Later, Khrapchenko [18] applied Rychkov's method (8) to estimate the complexity of the characteristic functions of BCH-codes.

Rychkov continued the study of Yablonskii's problem and proved the relation $L_{B_0}(l_n) \geq n^2 + 2 + (n \bmod 2)$ for $5 \leq n \neq 2^k$ [58]. The proof of the equalty $L_{B_0}(l_6) = 40$ obtained by D. Yu. Cherukhin [27] is very nontrivial, see also [59]. As a consequence, for $n \leq 8$ Yablonskii's formulas are optimal. In addition, S. V. Zdobnov [68] managed to show that Yablonskii's method is optimal for all n in a restricted class of formulas — which are represented by π-circuits without null chains.

M. Paterson (see [66]) proposed a generalization of the Khrapchenko method, the principle of formal complexity measures. A formal complexity measure is defined as a functional $\mu : (\mathbb{B}^n \to \mathbb{B}) \to \mathbb{R}_+$ with the properties: (1) $\mu(x_i) = 1$, $i = 1, \ldots, n$, (2) $\mu(\overline{f}) = \mu(f)$, (3) $\mu(f \vee g) \leq \mu(f) + \mu(g)$. It follows from the definition that $L_{B_0}(f) \geq \mu(f)$. The measure corresponding to Khrapchenko's method is

$$\mu(f) = \max_{N_0 \subset f^{-1}(0),\ N_1 \subset f^{-1}(1)} \frac{|R(N_0, N_1)|^2}{|N_0| \cdot |N_1|}.$$

Subsequently, several more complexity measures were proposed by different authors, but none of them allows obtaining superquadratic lower bounds, see [39].

Perhaps even more valuable than relation (7) is the way of reasoning in Khrapchenko's method. This is one of the first methods in the theory of complexity lower bounds based on the dual nature of zeros and ones of a function. E.g., in Khrapchenko's method, ones correspond to chains of a π-circuit, and zeros correspond to irredundant cuts. Chains and irredundant cuts are objects of dual nature. Subsequently, this type of argument became widespread: for example, the duality of zeros and ones underlies methods for obtaining complexity lower bounds for monotone circuits and bounded-depth circuits, see [39]. Essentially, the concept of communication complexity grows out of duality. So, it is not surprising that Khrapchenko's method admits a natural description in terms of communication complexity. This interpretation was proposed by M. Karchmer and A. Wigderson [40], who obtained an alternative proof of the analogue of (7) for the depth of formulas.

Along the path indicated by Andreev [24], formula complexity lower bounds of the form $n^{3-o(1)}$ have been obtained for specially constructed Boolean functions, see, e.g., [36, 64, 32]. However, for example, for symmetric Boolean functions, more than quadratic lower bounds are are still unknown.

The note by Khrapchenko [9] can also be attributed to the direction of complexity lower bounds. In it, he refined the result of E. Specker [38], showing that all symmetric functions of n variables, with the exception of 16 functions, have nonlinear formula complexity $\Omega(n\alpha(n))$ over an arbitrary complete basis, where $\alpha(n)$ is a very slowly growing function. At the same time, such a corollary stated for binary bases was obtained by M. Paterson [50]. Later, P. Pudlák [54] showed that the lower bound holds with $\alpha(n) = \log\log n$, and it is already tight in order. Nevertheless, D. Yu. Cherukhin [26] managed to prove that almost all symmetric functions have complexity $\Omega(n \log n)$ (but there are more than 16 functions-exceptions for this bound). The results [54, 26] are valid in any complete basis.

Formulas for symmetric Boolean functions.

It is natural that Khrapchenko became interested in the question of how hard it is to implement symmetric functions by formulas. Due to the uniformity of Boolean formulas, the question of depth is closely related to the question of the formula complexity.

Recall that the values of a symmetric Boolean function are determined by the arithmetic sum of its arguments. Symmetric functions play an important role in computing practice: the majority function and other simple threshold functions, periodic functions, and the bits of the Boolean operator C_n of summation of n bits are symmetric.

Let S_n denote the class of symmetric functions of n variables. Any function $f \in S_n$ can be represented as $h(C_n(x_1, \ldots, x_n))$, where h is some function of $\log_2 n$ arguments. It has been known since the early 1960s that $D_{B_0}(C_n) = O(\log n)$, see, e.g., [49]. When combining this result with the already known at that time methods of synthesis of arbitrary functions (the cascade method, asymptotically optimal methods, see [43, 48]), we immediately obtain the estimates $D_B(S_n) = O(\log n)$ and $L_B(S_n) = n^{O(1)}$ in any complete basis B.

Nevertheless, Khrapchenko's work [7] was the first in which the bound $L_B(S_n) = n^{O(1)}$ was stated explicitly (the author separately noted that the result is also valid in k-valued logic). But the main content of the work [7] were accurate estimates of the complexity of the operator C_n and the class S_n in the basis B_0: $L_{B_0}(C_n) = O(n^{4.62})$, $L_{B_0}(S_n) = O(n^{4.93})$. The complexity of threshold and elementary symmetric functions is estimated in the same way as the complexity of C_n.

To implement C_n, Khrapchenko applied the method of compressors [49]. A $(k,\, l)$-compressor is a circuit that transforms k numbers into $l < k$ numbers while preserving the sum. A multi-bit compressor is composed of parallel copies of single-bit compressors and therefore has depth $O(1)$, regardless of the length of the summands. Thus, using a tree of $(k,\, l)$-compressors, one can transform n summands into l summands with depth $O(\log n)$ and formula complexity $n^{O(1)}$. Khrapchenko constructed a special (7, 3)-compressor and described an efficient method for constructing a tree of such compressors.

The most important application of the compressor method is the implementation of multiplication of n-bit numbers; the main interest here is the depth of the computation. Indeed, by pairwise multiplication of bits, the product of integers is reduced to the summation of n numbers (the school method of multiplication). Therefore, for the depth of the operator M_n of multiplication of n-bit numbers, $D_B(M_n) \leq D_B(C_n) + D_B(\Sigma_n) + O(1)$ holds. Recall that Σ_n denotes the operator of summation of n-bit integers. In [10] Khrapchenko, using another design of the (7, 3)-compressor, obtained bounds $D_{B_0}(C_n) \lesssim 5.12 \log_2 n$ and, taking into account $D_{B_0}(\Sigma_n) \sim \log_2 n$ [3], the bound $D_{B_0}(M_n) \lesssim 6.12 \log_2 n$.

We note that later E. Grove in [35] (cited from [52]) proved that the difference between the depth of reduction of the summation arising from the school n-digit multiplication to the usual addition of two numbers and the depth of C_n actually is $o(\log n)$.

The work of Khrapchenko [7] was followed by a series of results by different authors with refinements of the upper bounds for $L_B(C_n)$ and $L_B(S_n)$, mainly in the basis $B = B_2$ of all binary Boolean functions. The intermediate results were summed up by M. Paterson, N. Pippenger and

U. Zwick [51, 52], who established an asymptotically optimal way of placing compressors in a circuit. As applied to Khrapchenko compressors, the new method yields the estimates $L_{B_0}(C_n) = O(n^{4.60})$, $L_{B_0}(S_n) = O(n^{4.85})$ and $D_{B_0}(C_n) \le (5.07 + o(1)) \log_2 n$. The authors [51, 52] also proposed several new techniques for designing and using compressors, and improved the above estimates to $L_{B_0}(C_n) = O(n^{4.57})$ and $D_{B_0}(C_n) \le (4.95 + o(1)) \log_2 n$.

I. S. Sergeev [61] proposed to compute the remainders $\sigma \bmod 2^k$ and $\sigma \bmod 3^l$ (the lowest components of the operator C_n and the analogous counting operator in the ternary number system) and the approximate value $\sigma^* \approx \sigma$ to finally calculate the sum of bits $\sigma = x_1 + \ldots + x_n$, obtaining the best currently known estimates: $L_{B_0}(C_n) = O(n^{3.91})$, $L_{B_0}(S_n) = O(n^{4.01})$, $D_{B_0}(C_n) \le (4.14 + o(1)) \log_2 n$, $D_{B_0}(S_n) \le (4.24 + o(1)) \log_2 n$.

Depth and delay of circuits.

Khrapchenko's latest series of papers is devoted to a curious problem that apparently no one had realized before him. It is about how accurately the depth of circuits corresponds to the concept of delay in the physical sense. Khrapchenko defined the delay $T(S)$ of a circuit S as the minimum time sufficient to establish the correct value at the circuit outputs for any values of the inputs, assuming that the delays of functional elements are equal to 1.

Despite the fact that $D_B(f) = T_B(f)$ always holds, the equality may not hold for a specific circuit, in particular for a minimal circuit implementing the function f. In [11, 13] Khrapchenko managed to construct a sequence of functions f_k such that for all minimal circuits S implementing them, $D_{B_0}(S) \sim 2T_{B_0}(S)$ holds.

In [16, 20] this result was strengthened. First [16], Khrapchenko constructed a sequence of minimal circuits S_k with the property $T_{B_0}(S_k) < \log_2 D_{B_0}(S_k) + 6$. In the final paper of the series [20] he removed the last restriction by presenting a sequence of functions f_k all minimal circuits S_k of which[2] satisfy $T_{B_0}(S_k) < \log_2 D_{B_0}(S_k) + 14$. This result is final due to the easily verified inequality: $T_{B_0}(S) \ge \log_2 D_{B_0}(S)$ for any minimal circuit S [16].

In [13] Khrapchenko also constructed an n-bit adder circuit over the standard basis with delay $(1 + o(1)) \log_2 n$ and complexity $(11 + o(1))n$. Thus, about n elements were saved when moving from the requirement of asymptotically minimal depth to an asymptotically minimal delay, see above.

These results of Khrapchenko open a new direction of research — minimization of the complexity of functions under a delay restriction. The problem is especially meaningful in cases where a similar depth restriction does not allow to construct a sufficiently simple circuit.

However, the study of delay in the sense of Khrapchenko has not yet re-

[2] In fact, each function f_k has a unique minimal circuit.

ceived further development. But if we look at it more broadly, the movement towards more accurate modeling of the physical characteristics of circuits, in the mainstream of which Khrapchenko's works are located, continues. For example, S. A. Lozhkin and B. R. Danilov [42] introduced a model of circuits in which the delay of functional elements depends on the values at their inputs, and obtained asymptotically accurate results on the depth of computation of Boolean functions in this model. Note that the circuit delay in this model depends on the values of inputs, as in the case of Khrapchenko's delay.

Conclusion.

V. M. Khrapchenko closely followed results in the area of complexity lower bounds and prepared a very detailed survey [15] for Kiberneticheskii Sbornik[3] in 1984. In his works [17, 23] he presented his own viewpoint of the work of two major specialists in synthesis and complexity theory, R. G. Nigmatullin and O. B. Lupanov – mathematicians whom Valery Mikhailovich knew well.

Khrapchenko's method of lower bounds for formula complexity has been included in many lecture courses and in all major monographs on the complexity of Boolean functions [48, 31, 39, 60, 66]. A description of Khrapchenko's parallel adder is given in books that focus on fast arithmetic [60, 66]. In books [31, 66], relation (6) is proved that connects the depth and complexity of formulas, but Khrapchenko's priority was not known to the authors.

Valery Mikhailovich was one of the main actors in the theory of synthesis and complexity in the most active years of the formation of this scientific direction in the second half of the 20th century. His results left a bright trace, as did his extraordinary personality, in the memory of all who knew him.

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