Complexity of boolean linear operators

I.S. Sergeev, 2021

Stasys Jukna

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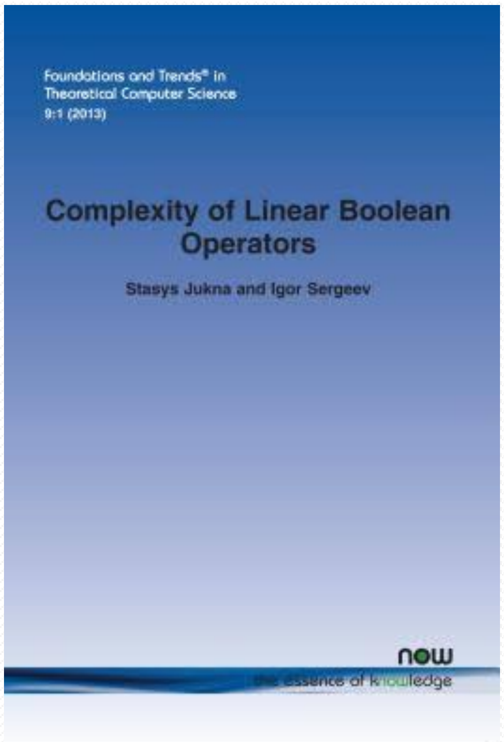
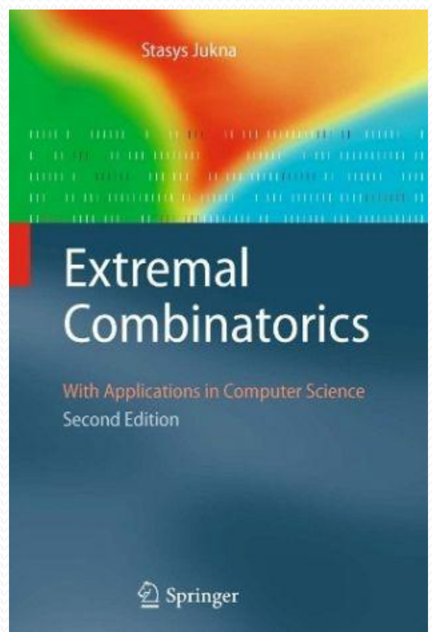
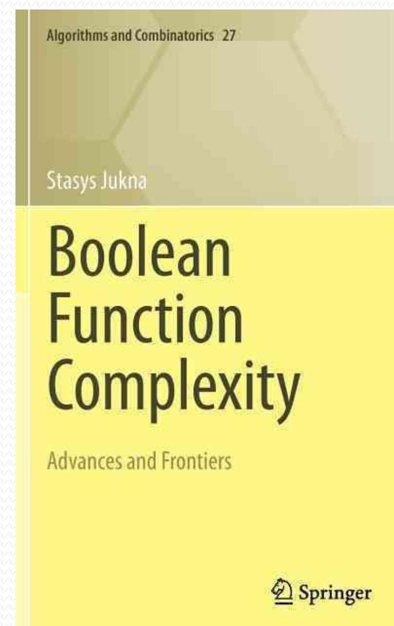
JOHANN WOLFGANG GOETHE UNIVERSITÄT FRANKFURT AM MAIN

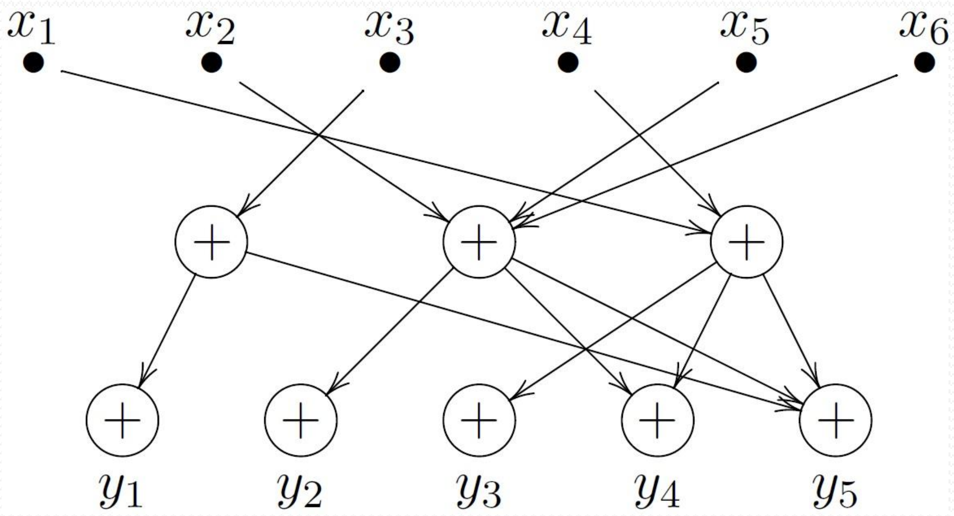


JS13





Linear circuits

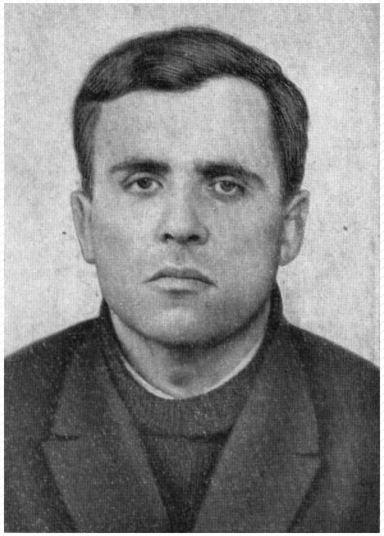


$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$p_{i,j} = \{\text{number of paths connecting } x_j \text{ and } y_i\}$$

SUM :	$(\mathbb{Z}_{\geq 0}, +)$	$A[i,j] = p_{i,j}$
OR :	$(\mathbb{B}, \vee)$	$A[i,j] = (p_{i,j} \geq 1)$
XOR :	$(\mathbb{B}, \oplus)$	$A[i,j] = p_{i,j} \bmod 2$

Asymptotic complexity bounds



O.B. Lupanov

E.I. Nechiporuk

$$L_2(n) \sim \frac{n^2}{\log n}$$

(1956)

$$L_3(n) \sim L(n) \sim \frac{n^2}{2 \log n}$$

(1963)

An upper bound via rank

Pavel Pudlák

Vojtěch Rödl

$$L_2(A) \preceq \mathrm{rk}(A) \cdot n, \qquad L_3(A) \preceq \frac{\mathrm{rk}(A) \cdot n}{\log n}$$

(1994)

Complexity of the recursively defined dmatrices

$$A_{2n} = \begin{bmatrix} f_1(A_n) & f_2(A_n) \\ f_3(A_n) & f_4(A_n) \end{bmatrix} \quad f_i(A) \in \{0,\ 1,\ A,\ \overline{A}\}$$

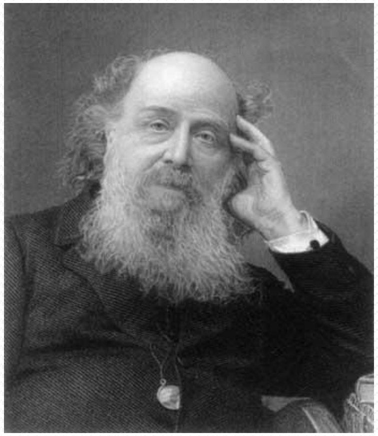
$$\mathsf{SUM}(A) \preceq n \log n$$

Intersections matrix:

$$K_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad K_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_{2n} = \begin{bmatrix} K_n & 1 \\ K_n & K_n \end{bmatrix}$$

$$rk_{\vee}(K) = \log n \qquad \mathsf{OR}_3(K) \asymp n$$

Sylvester-Hadamard matrices



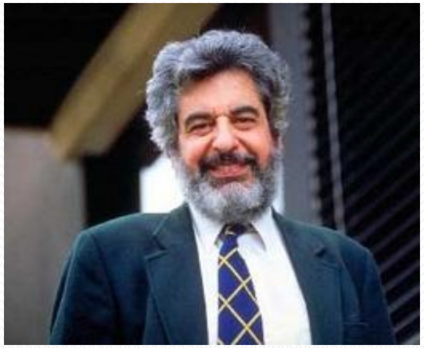
James Sylvester

Jacques Hadamard

$$H_1 = \begin{bmatrix} 0 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & \overline{H}_n \end{bmatrix}$$

$$rk_{\oplus}(H) = \log n \qquad\qquad |\det H^*| = 2(n/4)^{n/2}$$

Lower bounds via determinant



Jacques Morgenstern V.V. Kochergin

$$\mathsf{SUM}(A) \geq 3\log_3|\det(A)|$$

(1973, 2009)

$$\mathsf{SUM}_d(A) \geq dn|\det(A)|^{\frac{2}{dn}}$$

(1998)

$$SUM(H) \asymp n\log n, \qquad SUM_d(H) \asymp dn^{1+\frac{1}{d}}$$

Lower bounds via rigidity

Rigidity:

$$\mathrm{Rig}_A(r) = \min\{|B| : \mathrm{rk}(A \oplus B) \le r\}$$

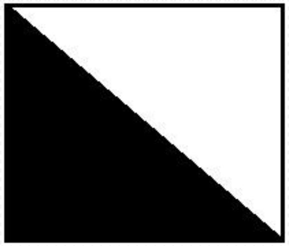
Leslie Valiant

T. $$\mathrm{Rig}_A(r) \ge \frac{f^2(n)}{r}, \qquad s \le r \le t$$

$$\Longrightarrow \quad \mathsf{XOR}_2(A) \ge 2f(n) \ln \frac{t}{s}$$

(1994)

Complexity of the full triangular matrix

$T =$ 

$$\mathrm{Rig}_T(r) \geq (1 - o(1))\frac{n^2}{4r}, \qquad r \in \omega(1) \cap o(n)$$

$$\mathsf{XOR}_2(T) \asymp n \log n$$

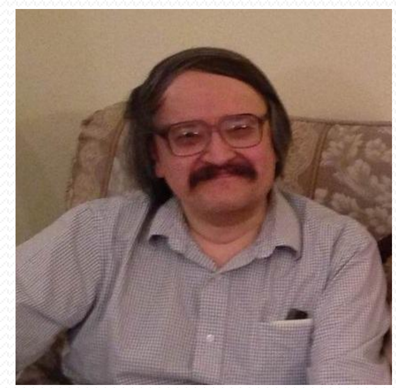
$$\mathsf{XOR}_d(T) \asymp n\lambda_d(n) \quad (1994)$$

$$\mathsf{OR}_d(T) \asymp n\lambda_d(n) \quad (1985)$$

Ashok Chandra

Steven Fortune

Richard Lipton



M.I. Grinchuk

Lower bound via the independent set cardinality

Georges Hansel

R.E. Krichevskii

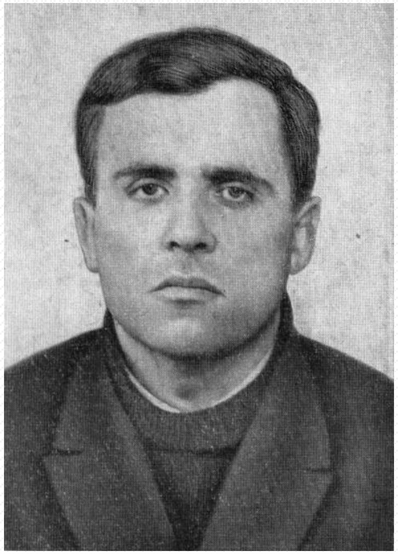
$$A \vee A^T = \overline{E} \quad \Longrightarrow \quad OR_2(A) \geq n \log n$$ (1964)

Tamás Tarján

$$\mathsf{OR}_2(T) \sim n \log n$$

$$\mathsf{OR}_2(K) \sim n \log n$$

Lower bounds via matrix weight



(k,l)-thin matrix:
does not contain size k x l rectangles (i.e. all-1 submatrices)

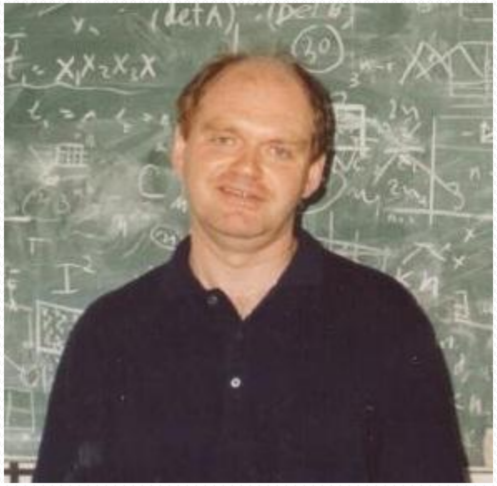
(1964)

T. A — $(k+1, l+1)$–thin matrix $\Longrightarrow$

$$\mathsf{OR}(A) \geq \frac{|A|}{k \cdot l} \qquad \mathsf{OR}_2(A) \geq \frac{|A|}{\max\{k, l\}}$$

* this final form of results is due to N. Pippenger (1980)

Lower bounds via matrix weight (2)



D.Yu. Grigoriev

$r(A)$ – maximal area of a rectangle in the matrix A

$$\mathsf{OR}(A) \geq \frac{3|A|}{r(A)} \log_3 \frac{|A|}{n}$$

(1976)

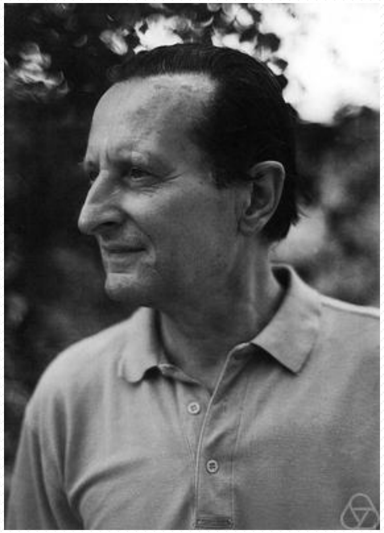
$$\mathsf{OR}_d(A) \geq \frac{d|A|}{r(A)} \left(\frac{|A|}{n}\right)^{1/d}$$

JS13

$$\mathsf{OR}(H) \asymp n \log n$$

$$\mathsf{OR}_d(H) \asymp dn^{1+1/d}$$

Kneser-Sierpinski matrix



Martin Kneser   Wacław Sierpiński

$$D_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad D_{2n} = \begin{bmatrix} D_n & 0 \\ D_n & D_n \end{bmatrix}$$

$$D = \overline{K}$$

Lower bounds for block matrices

S.N. Selezneva

Joan Boyar

Magnus Find

(2012)

$$L \in \{\mathsf{SUM},\ \mathsf{OR}\} \qquad M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

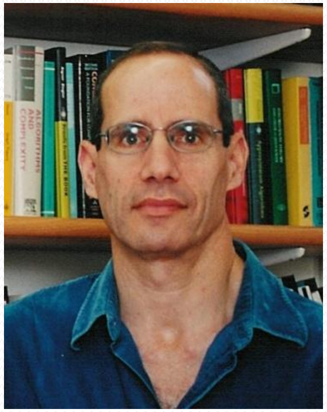
$$L(M) \geq L(A) + L(C) + \mathrm{rk}(B)$$

$$L_2(M) \geq L_2(A) + L_2(C) + \mathrm{tr}(B)$$

JS13

$$\mathsf{SUM}(D) \asymp \mathsf{OR}(D) \sim (1/2)n \log n$$

Combinatorial methods



Noga Alon

Mauricio Karchmer

Avi Wigderson

(1990)

$$dist(A) = \min_{i \neq j} |Ae_i - Ae_j|$$

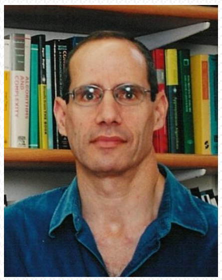
$$\mathsf{XOR}_2(A) \succeq dist(A) \cdot \frac{\log n}{\log \log n}$$

Andrew Drucker

$$B: \quad dist(B) \asymp n, \quad \mathsf{XOR}_2(B) \asymp n \cdot \frac{\log n}{\log \log n}$$

(2011)

Combinatorial methods (2)




Wolfgang Maass

m-Ramsey matrix: does not contain monochromatic (all-0 or all-1) rectangles of size *m* x *m*

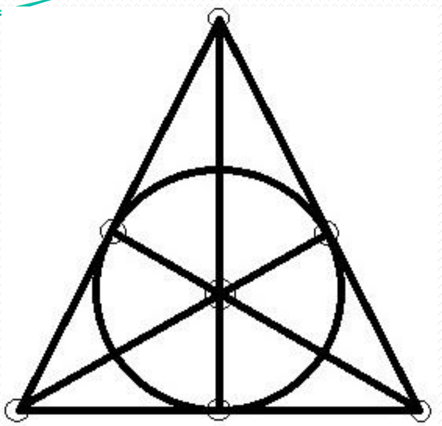
T. $A - n^c\text{-Ramsey matrix},\ c < 1$

$$\Longrightarrow \quad \mathsf{XOR}_2(A) \succeq n \log n$$

(1990)

$$\mathsf{XOR}_2(H) \asymp n \log n$$

Extremal matrices



$$S_7 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

James Singer

János Kollár Lajos Rónyai Tibor Szabó

(1996) $N[i,j] = ((\alpha_i - \alpha_j)^{\frac{q^t-1}{q-1}} = 1), \quad \alpha_i \in GF(q^t)$

N — $(t, t!+1)$–thin matrix, $|N| = q^{2t-1}$

OR/XOR separations

S.B. Gashkov I.S.

$$\mathsf{OR}(S)/\mathsf{XOR}(S) \succeq \frac{\sqrt{n}}{\log n \cdot 2^{O(\log^* n)}}$$

$$\mathsf{OR}(N)/\mathsf{XOR}(N) = n^{1-o(1)}$$

(2010-2014)

$$\mathsf{OR}_2(H)/\mathsf{XOR}_2(H) \asymp \frac{\sqrt{n}}{\log n}$$

T. A — random $n \times n$–submatrix of H_{n^2}

$$\Longrightarrow \quad \frac{\mathsf{OR}(A)}{\mathsf{XOR}_3(A)} \succeq \frac{\mathsf{OR}_2(A)}{\mathsf{XOR}_2(A)} \asymp \frac{n}{\log^2 n}$$

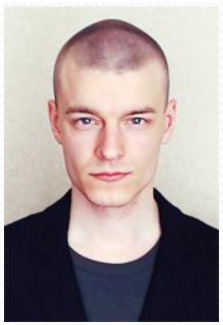
(2006) JS13

Lower bounds for Kronecker products

(1988)

$$L_2(B \otimes A) \geq \mathrm{tr}(B) \cdot L_2(A)$$

Anna Gal



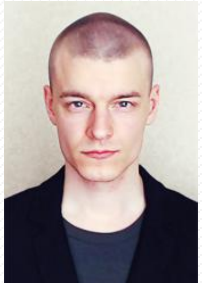
M.Find, M.Göös, M.Järvisalo, P.Kaski, M.Koivisto, J.Korhonen

T. A — $(k+1, l+1)$–thin matrix $\Longrightarrow$

$$L(B \otimes A) \geq \mathrm{rk}(B) \cdot \frac{|A|}{k \cdot l} \qquad L \in \{\mathsf{SUM},\ \mathsf{OR}\}$$

(2013)

SUM/OR separations

$$L_3(B \otimes A) \leq \mathrm{rk}(B) \cdot n^2 \qquad L_6(B \otimes A) \preceq \mathrm{rk}(B) \cdot \frac{n^2}{\log n}$$

$$M = \overline{E_{\sqrt{n}}} \otimes A_{\sqrt{n}}, \qquad A \text{ — random matrix}$$

$$\mathsf{SUM}(M)/\mathsf{OR}(M) \succeq \frac{\sqrt{n}}{\log^2 n}$$

(2013)

$$B: \quad \mathsf{OR}_2(B) \asymp n, \quad \mathsf{SUM}_2(B) \asymp n \log n$$

Trevor Pinto

OR-complexity of the complement matrix

Nets Katz

T. (2012)

(i) A — 2-thin matrix,

$|A| \succeq n^{1,1}$, $\mathrm{rk}(\overline{A}) \asymp \log n$

(ii) A — $\log n$-thin matrix,

$|A| \asymp n^2$, $\mathsf{OR}_2(\overline{A}) \preceq n \log^2 n$

JS13

$$\mathsf{OR}(A)/\mathsf{OR}(\overline{A}) \succeq \frac{n}{\log^3 n}, \qquad \mathsf{OR}_2(A)/\mathsf{OR}_2(\overline{A}) \succeq \frac{n}{\log^3 n}$$

Explicit bounds: $\frac{\mathsf{OR}(A)}{\mathsf{OR}(\overline{A})} = n^{1-o(1)}$ I.S.14 $\quad \frac{\mathsf{OR}_2(A)}{\mathsf{OR}_2(\overline{A})} \preceq n^{1/2-o(1)}$ JS13

SUM-complexity of the complement matrix

Kazuyuki Amano

Manami Shigeta

$$A: \quad A \vee A^T = \overline{E}, \quad \mathrm{rk}_+ A = n^{1/2+o(1)}$$

(2015)

$$M = \overline{A} \otimes B, \quad B \text{ — random matrix}$$

$$\mathsf{SUM}(M)/\mathsf{SUM}(\overline{M}) \succeq n^{1/4-o(1)}$$

JS21

Open problems (stated for $n \times n$ matrices)

- nonlinear lower bounds on XOR-complexity
- SUM-complexity of the matrix *K*: $n \preceq \mathsf{SUM}(K) \preceq n \log n$
- depth-2 complexity of the matrix *D*:

$$n^{1.16} \prec \mathsf{OR}_2(D) \leq \mathsf{SUM}_2(D) \prec n^{1.28}$$

JS13

$$\mathsf{OR}_2(D) \prec n^{1.17}$$

D. Chistikov, Sz. Iván, A. Lubiw, J. Shallit (2015)

- verify the rank conjecture: $\mathsf{L}(A \otimes B) \geq \mathrm{rk}(A) \cdot \mathsf{L}(B)$?
- does it hold that for a matrix A, $\mathsf{XOR}(A)/\mathsf{OR}(A) \to \infty$?
- construct a matrix *A* such that $\mathsf{SUM}_2(\overline{A})/\mathsf{SUM}_2(A) \to \infty$
- investigate L_2/L separations:

$$\frac{\mathsf{OR}_2(A)}{\mathsf{OR}(A)} \succeq \sqrt{n/\log n}, \qquad \frac{\mathsf{XOR}_2(B)}{\mathsf{XOR}(B)} \succ n^{0.3}$$

JS13-17