Note from 11.11.2026. In part VII of the proof of Theorem 3, it is not sufficient to verify the condition that the function $\varphi(x)$ is increasing on the interval $[t_\chi, (1-t_\chi)/2]$. A stronger condition is required: $\varphi(y) - \varphi(x) \geq 0$ on this interval. Recall that

$$\varphi(x) = \sqrt{x}\left(\left(\sqrt{x}+\sqrt{y}\right)^{\theta-1} + \left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1}\right),$$

where $y = 1 - t_\chi - x$, $\theta \approx 5.717$ and $t_\chi \approx 0.09314$.

The required inequality $\varphi(y) - \varphi(x) \geq 0$ can be verified directly as follows:

$$\varphi(y) - \varphi(x) = \left(\sqrt{y}-\sqrt{x}\right)\left(\left(\sqrt{x}+\sqrt{y}\right)^{\theta-1} + \left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1} + \left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta-1}\right) - \left(\sqrt{y}\left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1} - \sqrt{x}\left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta-1}\right). \quad (*)$$

Further, we apply Simpson's 1/3 rule interpolation formula — the inequality

$$\int_a^b f(z)dz \leq \frac{b-a}{6}\left(f(b) + 4f\left(\frac{a+b}{2}\right) + f(a)\right),$$

which holds provided that $f^{(4)}(z) \geq 0$ on $[a,b]$, by choosing the function $f(z) = z^{\theta-1}$ and setting $a = \sqrt{x} + \sqrt{t_\chi}$ and $b = \sqrt{y} + \sqrt{t_\chi}$. Now, we estimate the term being subtracted in $(*)$ for the case $y > x$:

$$\frac{\sqrt{y}\left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1} - \sqrt{x}\left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta-1}}{\sqrt{y}-\sqrt{x}} < \frac{\left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta} - \left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta}}{\sqrt{y}-\sqrt{x}} \leq$$
$$\leq \frac{\theta}{6}\left(\left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1} + 4\left(\frac{\sqrt{x}+\sqrt{y}}{2}+\sqrt{t_\chi}\right)^{\theta-1} + \left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta-1}\right) <$$
$$< \frac{2\theta}{3}\left(\frac{\sqrt{x}+\sqrt{y}}{2}+\sqrt{t_\chi}\right)^{\theta-1} + \left(\sqrt{y}+\sqrt{t_\chi}\right)^{\theta-1} + \left(\sqrt{x}+\sqrt{t_\chi}\right)^{\theta-1},$$

since $\theta < 6$. It remains to be seen that

$$\frac{2\theta}{3}\left(\frac{\sqrt{x}+\sqrt{y}}{2}+\sqrt{t_\chi}\right)^{\theta-1} \leq \left(\sqrt{x}+\sqrt{y}\right)^{\theta-1}.$$

Indeed,

$$\frac{2\theta}{3}\left(\frac{1}{2}+\frac{\sqrt{t_\chi}}{\sqrt{x}+\sqrt{y}}\right)^{\theta-1} \leq \frac{2\theta}{3}\left(\frac{1}{2}+\frac{\sqrt{t_\chi}}{\sqrt{t_\chi}+\sqrt{1-2t_\chi}}\right)^{\theta-1} \approx 0.999 < 1.$$